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The Noncommutative Dynamics and Topology of Iterated Function Systems

Alexander Don Munday

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The Noncommutative Dynamics and Topology of Iterated Function Systems

Alexander Don Munday

This thesis is presented as required for the conferral of the degree:

Doctor of Philosophy

Supervisor:
Associate Professor Adam Rennie

Co-supervisor:
Professor Aidan Sims

The University of Wollongong
School of Mathematics and Applied Statistics

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Declaration

I, *Alexander Don Munday*, declare that this thesis is submitted in fulfilment of the requirements for the conferral of the degree *Doctor of Philosophy*, from the University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. This document has not been submitted for qualifications at any other academic institution.

Alexander Don Munday

Tuesday 25th February, 2020

Abstract

Using operator algebraic techniques, we explore the relationship between the dynamics and topology of iterated function systems. We examine the C^* -algebras introduced by Kajiwara and Watatani, and extend many of their results to the non-contractive setting with large overlaps. We build upon their work with invertible systems to show that every Kajiwara-Watatani algebra is a subalgebra of an Exel crossed product. An investigation into the feasibility of a groupoid model for Kajiwara-Watatani algebras is undertaken, and in the process we develop novel topological techniques for groupoid C^* -algebras. Finally, we introduce a new C^* -algebra, called the lacunary algebra, which is built from an iterated function system. This algebra is sensitive to the interaction between the topology and dynamics of the system, and we compute its K -theory for an illustrative class of examples.

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Introduction

Motivation and History

The overall aim of this thesis is to analyse the interaction between the topology and dynamics of iterated function systems using noncommutative techniques from the field of operator algebras. Iterated function systems are discrete dynamical systems that have a strong relationship to fractal geometry. This thesis builds on the previous success that operator algebras have had studying topological dynamics.

Iterated function systems were first introduced by Hutchinson [Hut81]. Hutchinson proved that if Γ is a finite collection of strict contractions on a complete metric space X , then there is a unique non-empty compact set \mathbb{A} , such that

$$\mathbb{A} = \bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A}). \quad (1)$$

Moreover, for any compact subset K of X , repeated application of the *Hutchinson operator* $\Gamma(K) := \bigcup_{\gamma \in \Gamma} \gamma(K)$ produces a sequence of compact sets that converge to \mathbb{A} in the Hausdorff metric topology. The set \mathbb{A} is called the *attractor* of the *iterated function system* (X, Γ) . The attractor is often fractal in nature and can have non-integral Hausdorff dimension. The attractive nature of \mathbb{A} makes computational approximation of the attractor feasible by means of Barnsley's so-called *chaos game* [Bar93; BV11]. Due to their computability, iterated function systems have found a variety of applications including some applications to image compression [BH93].

Since their introduction iterated function systems have been generalised in a number of directions. An early extension by Barnsley and Demenko [BD85] was iterated function systems with probabilities. Another common generalisation is to forget the metric and let \mathbb{A} be a compact Hausdorff space. Then using the dynamics of a full shift space Ω_N , one can construct a collection of maps Γ on \mathbb{A} via the notion of a *code map* $\pi: \Omega_N \rightarrow \mathbb{A}$ (cf. [Kam00; Kig01]). Kieninger [Kie02] considers a general class of iterated function systems on compact Hausdorff spaces where Γ can consist of uncountably many maps. A survey of the iterated function system literature has been compiled by Barnsley and Vince [BV13].

In this thesis, we are primarily interested in the interplay between the dynamics induced by an iterated function system and the topology of the attractor \mathbb{A} . As we will see, small changes to the maps in Γ can result in significant changes to the topology of the attractor (see Example 1.3.6). We are therefore interested in invariants for iterated function systems that are built from both the collection of maps Γ and the attractor \mathbb{A} .

This is where operator algebras—and in particular, C^* -algebras—come into play. The field of operator algebras has a long and interwoven history with dynamical systems and

ergodic theory, dating back to the quantum mechanical origins of the field. Within a C^* -algebra it is often possible to encode both a topological space—as a commutative subalgebra—and a collection of operators that act on the commutative subalgebra, performing the desired dynamics. By computing properties of the associated “crossed product” C^* -algebra, one can frequently deduce properties of the original dynamical system. We mention some examples below.

There are many ways to build a C^* -algebra from a dynamical system. For iterated function systems we will use C^* -correspondences and their Cuntz-Pimsner algebras. Cuntz-Pimsner algebras were introduced by Pimsner [Pim97] and further expanded upon by Muhly and Solel [MS98] as well as Katsura [Kat04b; Kat07]. In essence, the maps implementing the dynamics are encoded within a C^* -correspondence. The associated Cuntz-Pimsner algebra is then a universal C^* -algebra which encodes the structure of the C^* -correspondence. An advantage of Cuntz-Pimsner algebras over other approaches is that their invariants (e.g. K -theory, KMS-states) are well-understood and frequently computable in terms of the underlying dynamics. Cuntz-Pimsner algebras also generalise many other dynamical C^* -algebras, including Cuntz-Krieger algebras [CK80], directed graph C^* -algebras [EW80; Kum+97], and crossed products by an automorphism.

Using Cuntz-Pimsner algebras to study iterated function systems is well-established. The first approach was by Pinzari-Watatani-Yonetani [PWY00]. However, the Cuntz-Pimsner algebra they considered is always isomorphic to the Cuntz algebra $\mathcal{O}_{|\Gamma|}$. Accordingly, the only information that can be extracted from the algebra itself is the number of maps in the system. There is no immediate way to detect information about the topology of the attractor \mathbb{A} or the nature of the maps in Γ .

Kajiwara and Watatani [KW06; KW14; KW16; KW17] took a different approach. They instead considered a C^* -correspondence built from the graph $\text{Gr}(\Gamma) = \bigcup_{\gamma \in \Gamma} \text{Gr}(\gamma)$ of the iterated function system. In the case of contractive systems satisfying the *open-set condition*, they showed that many invariants (K -theory, KMS-states, traces) of the associated Cuntz-Pimsner algebra $C^*(\mathbb{A}, \Gamma)$ depend on the *branching* (or *ramification*) structure of the system (\mathbb{A}, Γ) . In particular, if (\mathbb{A}, Γ) contains points $x \in \mathbb{A}$ for which there exists $y \in \mathbb{A}$ and $\gamma \neq \gamma' \in \Gamma$ with $x = \gamma(y) = \gamma'(y)$, then the associated Cuntz-Pimsner algebra $C^*(\mathbb{A}, \Gamma)$ is frequently not a Cuntz algebra. Unfortunately, if (\mathbb{A}, Γ) has no branching—which is often the case—then $C^*(\mathbb{A}, \Gamma)$ is again isomorphic to the Cuntz algebra $\mathcal{O}_{|\Gamma|}$.

De Castro [dCa09] has also worked on Kajiwara and Watatani’s algebras, and Ionescu and Watatani [IW08] have generalised them to the graph directed systems of Mauldin and Williams graphs [MW88]. Recently, Dor-On [DO18] has considered *weighted partial systems* and related their dynamics to various notions of isomorphism between non-self-adjoint tensor algebras.

There have also been other operator algebraic approaches to iterated function systems. Ionescu and Kumjian [IK14] have considered the C^* -algebra of a groupoid built from the action of a Deaconu-Renault groupoid on a so-called fractafold bundle. This groupoid algebra has not been particularly well-studied since its inception, and so the invariants of this algebra are not well known. Jorgensen and others have also studied the interplay between iterated function systems, wavelet analysis, and representations of Cuntz algebras [DJ06; DJ07b; DJ07a; Jor06; BJ99].

An iterated function system (\mathbb{A}, Γ) induces a natural action of the free monoid $\mathbb{F}_{|\Gamma|}^+$ on the algebra of continuous functions $C(\mathbb{A})$ by $*$ -endomorphisms (see Remark 2.1.5). Thus, possible approaches include semigroup crossed-products [LR96; Exe08; KL09], semigroup

C^* -algebras [Li12], or Cuntz-Nica-Pimsner algebras of product systems [SY10]. From the author’s experience, each of these approaches—when they apply—tends to ignore the topology of the attractor \mathbb{A} , and once again yields a Cuntz algebra.

There have also been many approaches to studying general fractals using the techniques of noncommutative geometry (cf. [KL01; GI03; GI05; GI16; CIL08; CIS12; Cip+14; GI17]). However, many of these approaches are concerned with the geometry of the fractal itself, and not the iterated function system for which the fractal is the attractor. Since inception, noncommutative geometry has allowed for the study of the geometry of spaces that are beyond the classical tools of differential geometry. Many approaches to studying fractals using noncommutative geometry involve building a spectral triple for an algebra of “smooth” functions on the fractal. Spectral triples—introduced by Connes [Con94]—are a noncommutative generalisation of smooth functions on a Riemannian manifold, together with the Dirac operator acting on L^2 -sections of a Clifford bundle. Spectral triples find applications in a variety of non-classical settings, including fractals.

The first example of a fractal spectral triple was given by Connes [Con94] on the middle thirds Cantor set. Connes’ ideas were further extended to build spectral triples on fractals built from curves [CIL08] and the Sierpinski gasket [CIS12]. For such classes of spectral triple, it is possible to recover the fractal as well as the Hausdorff measure, the Hausdorff dimension, and the metric of the fractal. In [Cip+14] it was shown that the Dirichlet form on the Sierpinski gasket—introduced by Kigami [Kig95; Kig01]—can also be recovered with the techniques of noncommutative geometry. These analytic techniques do not address the interplay of the topology and dynamics of iterated function systems.

The primary motivation of this thesis is to find a suitable C^* -algebraic construction which encapsulates both the dynamics and topology of an iterated function system, in such a way that invariants of the algebra are computable, and describe useful properties of the original system.

Thesis Structure and Results

We now outline the structure of the thesis and highlight the main results.

Chapter 1 We delve into the necessary background on iterated function systems. This includes a summary of Hutchinson’s Theorem (Theorem 1.1.4) for contractive iterated functions in Section 1.1. In Section 1.2, we introduce code maps, as well as the critical and post-critical sets of an iterated function system. We examine the relationship that the critical set has with the topology of the attractor. We also present numerous of foundational examples, which will be referred to throughout the thesis.

Most operator algebraic approaches to analysing iterated function systems assume that the system is contractive and has small critical sets, in the sense that it satisfies the open-set condition. We do not make either of these assumptions.

We finish Chapter 1 with a discussion of morphisms of iterated function systems. In Corollary 1.3.8, we present a new result that the graph of a morphism between two contractive systems is itself the attractor of an iterated function system.

Chapter 2 We generalise the Cuntz-Pimsner algebras considered by Kajiwara and Watatani [KW06], hereafter referred to as *Kajiwara-Watatani algebras*. We work in the setting of iterated function system satisfying Definition 1.0.1. In particular, we do not

assume that our systems are contractive, nor do we assume that the open-set condition is satisfied. Occasionally, we assume that our systems admit a code map.

We begin the chapter by revisiting the Cuntz-Pimsner algebra introduced by Pinzari-Watatani-Yonetani [PWY00] in our general setting. In Proposition 2.1.4, we give a new proof that Pinzari-Watatani-Yonetani’s algebra is isomorphic to a Cuntz algebra. Our proof does not rely on the underlying iterated function system being contractive, but instead on the weaker assumption of a code map existing.

In Section 2.2 we examine Kajiwara-Watatani algebras themselves. One of the main outcomes of this thesis has been the extension of Kajiwara and Watatani’s results to compact Hausdorff spaces, and the removal of the open-set condition hypothesis. In Definition 2.2.10, we give a new formulation of branched points for non-contractive systems. The definition is key to describing the covariance ideal associated to general Kajiwara-Watatani algebras (see Proposition 2.2.19).

In Section 2.3 we compare Kajiwara-Watatani algebras to the algebra previously considered by Pinzari-Watatani-Yonetani and interpret their differences in terms of the path spaces they encode. In Section 2.4, we compute the K -theory of some illustrative classes of Kajiwara-Watatani algebras, including some that do not satisfy the open-set condition.

In Section 2.5 we consider invertible iterated function systems in the sense of Kieninger [Kie02, Definition 5.4.6] and how invertibility of an iterated function system manifests in the corresponding Kajiwara-Watatani algebra. In Proposition 2.5.29, we show that every Kajiwara-Watatani algebra embeds in the Kajiwara-Watatani algebra of an invertible system. This is done using the inverse lifted system of Kieninger [Kie02, Definition 5.2.3] together with intermediary systems build on the graphs of iterated function systems that we call *graph systems* (see Definition 2.5.11). In Proposition 2.6.10, we show—using machinery developed by Kwaśniewski [Kwa17]—that the Kajiwara-Watatani algebra of an invertible system can always be realised as an Exel crossed product, generalising a result of de Castro [dCa09].

We finish Chapter 2 by extending the Hilbert module frame constructed by Kajiwara and Watatani [KW04] to a larger class of non-contractive iterated function systems, which do not necessarily satisfy the open-set condition.

Chapter 3 We investigate the feasibility of constructing a groupoid model for Kajiwara-Watatani algebras. We do this in the more general context of *topological quivers* [MS98; MT05b]. Topological quivers are a “non-étale” generalisation of the *topological graphs* introduced by Katsura [Kat04a]. A groupoid model for topological graphs is well-known, but because topological quivers have branched points, the existence of groupoid model is subtle in this case.

In Section 3.1 we introduce topological quivers along with their associated C^* -correspondences X_E and C^* -algebras \mathcal{T}_{X_E} and \mathcal{O}_{X_E} . We outline how both topological graphs algebras and Kajiwara-Watatani correspondences fit into the framework of topological quivers. We also recall the *Deaconu-Renault groupoids* associated to a topological graph.

In Section 3.2 we take a detailed look at Katsura’s [Kat04b] analysis of the core of a Cuntz-Pimsner algebra. With Katsura’s description of the core as a direct limit, we outline a strategy for building a groupoid model for the core of a topological quiver algebra. Following this programme, in Proposition 3.2.14, we produce a non-étale equivalence relation whose C^* -algebra is isomorphic to the compact operators $\text{End}_A^0(X_E)$ on X_E .

Continuing our programme we study this groupoid in more detail. We give a groupoid-theoretic description of the map $T \mapsto T \otimes \text{id}$ from $\text{End}_A^0(X^{\otimes n} \cdot I_{X_E})$ to $\text{End}_A^0(X^{\otimes n+1})$ in Proposition 3.2.28, where I_{X_E} is the covariance ideal of X_E .

In Section 3.3.1 we give a new construction of the *path space* and *boundary path space* of a topological graph as an inverse limit, which is also defined for topological quivers. The novelty in this construction is that it is built from the bottom-up, without reference to an ambient groupoid or C^* -algebra, as is usually the case. To do this, we employ a new topological construction known as perfection, which is described in Appendix C.3.

We finish Chapter 3 by giving a new, bottom-up, systematic construction of the core groupoids associated to a topological graph. This culminates in Theorem 3.3.35 and Theorem 3.3.43. In the process we introduce another novel topological technique: *adjunctions of groupoids*. Adjunctions of groupoids are examined in Appendix C.2.

Along the way, we highlight precisely where difficulties arise in the case of a general topological quiver (see Remark 3.3.18 and Remark 3.3.22). Although we only recover a groupoid in the known case of topological graphs, we have laid the foundations for constructing more general dynamical groupoids in the future.

Chapter 4 We introduce a new C^* -algebra associated to an iterated function system called the *lacunary algebra*. The lacunary algebra arises from a modification of the Cuntz-Pimsner algebra considered by Pinzari-Watatani-Yonetani.

In Section 4.1 we identify a subset of the post-critical set of an iterated function system, called the *singular boundary* (see Definition 4.1.5). In a sense that is made precise, points in the singular boundary act as an obstruction to continuous invertibility of the iterated function system. We modify the C^* -correspondence of Pinzari-Watatani-Yonetani by removing points arising from the singular boundary. The lacunary algebra is the Cuntz-Pimsner algebra of the resulting correspondence, and is introduced in Section 4.2.

In Section 4.3 we compute the K -theory of the lacunary algebra for an illustrative class of examples. These preliminary calculations indicate that the lacunary algebra is more sensitive to interplay between the dynamics and topology of an iterated function than either of the algebras considered by Pinzari-Watatani-Yonetani and Kajiwara-Watatani.

We finish Chapter 4 by examining the relationship between critical points and inner products on Hilbert modules. We introduce the *critical boundary* (see Definition 4.4.5) of an iterated function system as a closed subset of the critical set. This culminates in Theorem 4.4.9, which states that critical boundary is the obstruction to Pinzari-Watatani-Yonetani's correspondence being a bi-Hilbertian C^* -bimodule.

Appendices

Appendix A We revise the necessary background on Hilbert modules, including frames. We define C^* -correspondences and morphisms between them. We finish by introducing the Toeplitz algebra and Cuntz-Pimsner algebra of a C^* -correspondence.

Appendix B We define groupoids and their C^* -algebras. We introduce Haar systems and amenability of groupoids. Finally, we consider groupoid actions and corresponding transformation groupoids.

Appendix C We describe in detail the two new topological constructions which are required in Chapter 3.

The first construction is that of a *perfection*. A perfection extends the domain of a continuous map $p: X \rightarrow Y$ to arrive at a perfect map (a continuous, proper, surjection). A similar concept known as *fibrewise compactifications* have appeared before [Why66; Jam89; AD14], but do not make the resulting map surjective, only proper. We introduce both the *unified space* and *minimal perfection*. These are used to reconstruct the path space and boundary path space of a topological graph in Section 3.3.1.

The second new construction we introduce are *adjunction groupoids*. An adjunction groupoid—like an adjunction space—is the result of gluing two groupoids over a common subgroupoid. In Theorem C.2.10 we prove that under fairly relaxed hypotheses, the adjunction of two topological groupoids is a topological groupoid. If the original groupoids are étale, then so is the resulting groupoid. We use adjunction groupoids extensively in Section 3.3 to reconstruct the core groupoids for topological graphs.

Appendix D Appendix D contains a result about a certain form of commuting diagram for Abelian groups.

CHAPTER 1

Iterated Function Systems

We begin by introducing the central objects of this thesis, *iterated function systems*. Iterated function system were first introduced by Hutchinson [Hut81]. Hutchinson was interested in studying the dynamics of finite families of contractions on complete metric spaces. He proved that such collections of contractions have a unique attractor and possess a unique invariant measure. These results are now collectively known as Hutchinson’s Theorem (see Theorem 1.1.4) and they underpin the field of iterated function systems.

The term *iterated function system* was first used by Barnsley and Demenko [BD85] to describe a finite collection of Borel-measurable maps on a compact metric space which are compatible with a certain Markov-type operator. It is now more common to use the term iterated function system to refer to a family of continuous maps on either a metric or topological space. The iterated function system literature is extensive, and we could not hope to give a detailed account of the state of the field. We instead give a basic overview of iterated function systems and suggest [Kig01; Bar93; Bar06; Fal86; Edg08; BV13] for further reading.

Since we will use C^* -algebraic machinery to analyse iterated function systems, we are mostly be interested in the topological structure rather than the metric or measure-theoretic structure. Various definitions of iterated function systems exist throughout the literature. For the purposes of this thesis we choose to work with the following slightly non-standard definition.

Definition 1.0.1. An *iterated function system* (sometimes just a *system*) is a pair (\mathbb{A}, Γ) consisting of a second-countable compact Hausdorff space \mathbb{A} —called the *attractor*—and a finite collection Γ of continuous maps on \mathbb{A} . We require that the space \mathbb{A} is Γ -invariant in the sense that

$$\bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A}) = \mathbb{A}. \quad (1.1)$$

We say that (\mathbb{A}, Γ) is *injective* if each $\gamma \in \Gamma$ is injective. The Γ -invariance condition can be interpreted as saying that the maps $\gamma \in \Gamma$ jointly surject onto \mathbb{A} .

In some regards, the definition of an iterated function system that we have chosen subverts one of the main results in the theory, Hutchinson’s Theorem. However, it is similar to the definitions of the systems found in [Kig01] and [Kam00]. A somewhat more classical definition of an iterated function system is given in Definition 1.1.3.

Many authors (cf. [Kig01; Kam00]) assume that an iterated function system comes equipped with a continuous surjection from a full shift space, called *code map* (see Section 1.2), and we pay particular attention to this situation. Kieninger [Kie02], on the

other hand, deals with a more general case, where Γ is infinite but compact with respect to the compact-open topology.

1.1 | Contractive iterated functions systems

In this section we give an overview of the theory of contractive iterated function systems, first introduced by Hutchinson [Hut81]. To begin, we recall the definition of a contraction.

Definition 1.1.1. Let (X, d) be a metric space. A map $\gamma : X \rightarrow X$ is said to be a *contraction* if there exists $0 \leq K < 1$ such that

$$d(\gamma(x), \gamma(y)) < Kd(x, y)$$

for all $x, y \in X$. We say that a contraction γ is a *strict* if γ is injective.

It is well-known that a contraction map on a complete metric space has a unique fixed point. This result is commonly known as the Banach Fixed-Point Theorem.

Theorem 1.1.2 (Banach Fixed-Point Theorem (cf. [Kig01, Theorem 1.1.2])). *Let $\gamma : X \rightarrow X$ be a contraction on a non-empty complete metric space (X, d) . Then there is a unique $x_0 \in X$ such that $\gamma(x_0) = x_0$. For any $x \in X$ the sequence $(\gamma^n(x))_{n \in \mathbb{N}}$ converges to x_0 .*

Definition 1.1.3. A *contractive iterated function system* is a pair (X, Γ) consisting of a non-empty complete metric space (X, d) (we often suppress the metric d), together with a finite collection $\Gamma = \{\gamma_1, \dots, \gamma_N\}$ of strict contractions on X .

As it stands, a contractive iterated function system is not necessarily an iterated function system in the sense of Definition 1.0.1. However, the following fundamental theorem of contractive systems asserts that a contractive iterated function system gives rise to a unique iterated function system in the sense of Definition 1.0.1.

Theorem 1.1.4 (Hutchinson's Theorem [Hut81, §1]). *Let (X, Γ) be a contractive iterated function system. Then there is a unique non-empty compact subset $\mathbb{A} \subseteq X$ which is Γ -invariant in the sense that*

$$\mathbb{A} = \bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A}). \quad (1.2)$$

Definition 1.1.5. We call the set \mathbb{A} of Theorem 1.1.4 the *attractor* of (X, Γ) . In which case we say that (\mathbb{A}, Γ) is the iterated function system associated to the contractive system (X, Γ) .

We include a short overview of the proof of Hutchinson's theorem (based on the proof found in [Kig01]) because it illuminates the nature of the dynamics of a contractive iterated function system. In particular, it explains why the attractor is named as such.

To begin, let $\mathbb{H}(X)$ denote the collection of non-empty compact subsets of X . We call $\mathbb{H}(X)$ the *hyperspace* of X . By abuse of notation we write Γ for the map $\Gamma : \mathbb{H}(X) \rightarrow \mathbb{H}(X)$ given by

$$\Gamma(K) = \bigcup_{\gamma \in \Gamma} \gamma(K). \quad (1.3)$$

The map Γ is often called the *Hutchinson operator* due to its pivotal role in the proof of Hutchinson's theorem.

Let $d_H : \mathbb{H}(X) \times \mathbb{H}(X) \rightarrow [0, \infty)$ denote the *Hausdorff metric* on $\mathbb{H}(X)$. That is, for $K, K' \in \mathbb{H}(X)$ we have

$$d_H(K, K') := \inf\{\varepsilon \geq 0 \mid K \subseteq K'_\varepsilon \text{ and } K' \subseteq K_\varepsilon\},$$

where $K_\varepsilon := \{x \in X \mid d(x, K) \leq \varepsilon\}$ is the closed ε -neighbourhood of K . The proof of Hutchinson's Theorem relies on the following two results.

Proposition 1.1.6 ([Kig01, Proposition 1.1.5]). *The map $d_H : \mathbb{H}(X) \times \mathbb{H}(X) \rightarrow [0, \infty)$ defines a metric on $\mathbb{H}(X)$. Moreover, (X, d) is complete if and only if $(\mathbb{H}(X), d_H)$ is complete.*

Proposition 1.1.7 ([Kig01, cf. Theorem 1.1.7]). *The Hutchinson operator $\Gamma : \mathbb{H}(X) \rightarrow \mathbb{H}(X)$ is a contraction mapping with respect to d_H .*

Hutchinson's Theorem is now a simple consequence of these two results, together with the Banach Fixed-Point Theorem.

Proof of Theorem 1.1.4. As (X, d) is complete, Proposition 1.1.6 implies that $(\mathbb{H}(X), d_H)$ is complete. Proposition 1.1.7 says that Γ is a contraction mapping. The Banach Fixed-Point Theorem now implies that Γ has a unique fixed point \mathbb{A} . Since $\mathbb{A} \in \mathbb{H}(X)$ satisfies (1.2) if and only if it is a fixed-point for Γ , the result follows. \square

The use of the Banach Fixed-Point Theorem in the proof of Hutchinson's Theorem has the following important consequence.

Corollary 1.1.8. *Let (X, Γ) be a contractive iterated function system. Then for any compact set $K \subseteq X$ the sequence $\{\Gamma^n(K)\}_{n \in \mathbb{N}}$ converges to \mathbb{A} in the Hausdorff metric.*

Corollary 1.1.8 justifies calling the invariant set \mathbb{A} the attractor. Moreover, it gives a concrete way of approximating the attractor of a given contractive iterated function system: start with a compact set K (for example a point) and repeatedly apply the Hutchinson operator Γ .

A unique attractor for (X, Γ) is guaranteed so long as there exists some metric d on X for which every $\gamma \in \Gamma$ is a contraction. On the other hand, there exists complete metric spaces (X, d) , endowed with collections of continuous injections Γ , such that there exists a unique compact Γ -invariant set, but there is no metric on X which induces the same topology as d , under which each $\gamma \in \Gamma$ is a contraction (see [BV11, §4]).

Before we proceed we describe some fundamental examples.

Example 1.1.9. Let $X = \mathbb{R}$ with the Euclidean metric. Define $\Gamma = \{\gamma_1, \gamma_2\}$, where $\gamma_1(x) = \frac{x}{3}$ and $\gamma_2(x) = \frac{x+2}{3}$. These maps are clearly contractions. Let $S = [0, 1]$. Applying the Hutchinson operator Γ to S we see that $\Gamma(S) = [0, 1/3] \sqcup [2/3, 1]$, and $\Gamma^2(S) = [0, 1/9] \sqcup [2/9, 1/3] \sqcup [2/3, 7/9] \sqcup [8/9, 1]$. Further iterations are pictured in Figure 1.1. Continuing this process it is not hard to deduce that $\Gamma^n(S)$ converges to the middle-thirds Cantor set, which we denote by C . Indeed, we have $\Gamma(C) = C$, so Hutchinson's Theorem implies that C is the attractor. \triangle



Figure 1.1: The result of applying the Hutchinson operator from Example 1.1.9 to $S = [0, 1]$.

Example 1.1.10. Let $X = \mathbb{R}^2$ with the Euclidean metric. Consider the contractive iterated function system $(X, \Gamma = \{\gamma_1, \gamma_2, \gamma_3\})$ with maps given by,

$$\gamma_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right), \quad \gamma_2(x, y) = \left(\frac{x+1}{4}, \frac{2y+\sqrt{3}}{4}\right), \quad \text{and} \quad \gamma_3(x, y) = \left(\frac{x+1}{2}, \frac{y}{2}\right).$$

Each map scales by $\frac{1}{2}$ and then shifts according to the vertices of an equilateral triangle. We call the attractor of this iterated function system the *Sierpinski gasket* (see Figure 1.2).

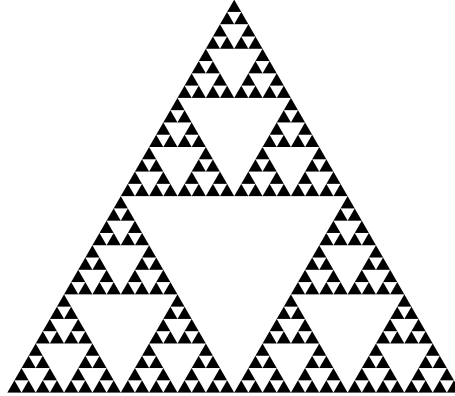


Figure 1.2: An approximation of the Sierpinski gasket; the attractor of (X, Γ) from Example 1.1.10.

Example 1.1.11. Let $X = \mathbb{R}^2$ with the Euclidean metric. Consider the contractive iterated function system $(X, \Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8\})$, with maps (laid out, suggestively, in a grid) given by

$$\begin{aligned} \gamma_6(x, y) &= \left(\frac{x}{3}, \frac{y}{3}\right) + \left(0, \frac{2}{3}\right) & \gamma_7(x, y) &= \left(\frac{x}{3}, \frac{y}{3}\right) + \left(\frac{1}{3}, \frac{2}{3}\right) & \gamma_8(x, y) &= \left(\frac{x}{3}, \frac{y}{3}\right) + \left(\frac{2}{3}, \frac{2}{3}\right) \\ \gamma_4(x, y) &= \left(\frac{x}{3}, \frac{y}{3}\right) + \left(0, \frac{1}{3}\right) & & & \gamma_5(x, y) &= \left(\frac{x}{3}, \frac{y}{3}\right) + \left(\frac{2}{3}, \frac{1}{3}\right) \\ \gamma_1(x, y) &= \left(\frac{x}{3}, \frac{y}{3}\right) & \gamma_2(x, y) &= \left(\frac{x}{3}, \frac{y}{3}\right) + \left(\frac{1}{3}, 0\right) & \gamma_3(x, y) &= \left(\frac{x}{3}, \frac{y}{3}\right) + \left(\frac{2}{3}, 0\right). \end{aligned}$$

We call the attractor of this system the *Sierpinski carpet* (see Figure 1.3).



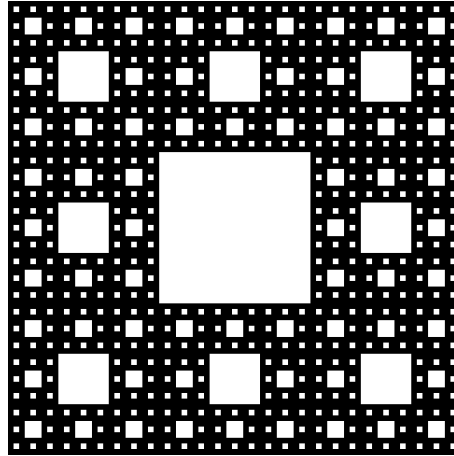


Figure 1.3: An approximation of the Sierpinski carpet; the attractor of (X, Γ) from Example 1.1.11.

Hutchinson also showed the existence of a unique invariant measure, usually called the *Hutchinson measure* or *self-similar measure* of an iterated function system. The Hutchinson measure is invariant in the sense of the following theorem.

Theorem 1.1.12 ([Hut81]). *Let $(X, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ be a contractive iterated function system X with attractor \mathbb{A} . Suppose that $r_1, \dots, r_N \in (0, 1)$ satisfy $\sum_{i=1}^N r_i = 1$. Then there exists a unique regular Borel probability measure μ on X such for all Borel sets $E \subseteq X$,*

$$\mu(E) = \sum_{i=1}^N r_i \mu(\gamma_i(E)).$$

Many authors restrict their attention to iterated function systems that satisfy a separation condition known as the open-set condition. The open-set condition goes back to Moran [Mor46] who used it to compute the Hausdorff dimension of bounded sets in \mathbb{R}^n .

Definition 1.1.13. Let $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ be an iterated function system with attractor \mathbb{A} . Then (\mathbb{A}, Γ) satisfies the *open-set condition* if there exists a non-empty set $V \subseteq \mathbb{A}$, such that

$$\bigcup_{i=1}^N \gamma_i(V) \subseteq V \quad \text{and} \quad \gamma_i(V) \cap \gamma_j(V) = \emptyset \text{ for all } i \neq j.$$

One of the benefits of iterated function systems satisfying the open-set condition is that the Hausdorff dimension of the attractor is often computable. Let (X, d) be a metric space. Recall that a function $\gamma: X \rightarrow X$ is a *similarity* if there exists $r > 0$ such that $d(\gamma(x), \gamma(y)) = rd(x, y)$ for all $x, y \in X$.

Theorem 1.1.14 (Moran's Theorem [Kig01, Corollary 1.5.9]). *Let (\mathbb{R}^n, d) be Euclidean space. Suppose that $(\mathbb{R}^n, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ is a contractive iterated function system consisting of similarities with $d(\gamma_i(x), \gamma_i(y)) = r_i d(x, y)$ for all $x, y \in \mathbb{R}^n$ and $1 \leq i \leq N$. Let \mathbb{A} be the attractor of (\mathbb{R}^n, Γ) , and suppose that (\mathbb{A}, Γ) satisfies the open-set condition. Let D be the unique positive number for which $\sum_{i=1}^N r_i^D = 1$ (called the *similarity dimension* of (\mathbb{A}, Γ)). Then D is equal to the Hausdorff dimension of \mathbb{A} .*

Theorem 1.1.14 shows that the attractor of an iterated function system often does not have integral Hausdorff dimension. Theorem 1.1.14 has also been generalised by Kigami to situations outside of Euclidean space (see [Kig01, Theorem 1.5.7]).

If we use Mandelbrot's original definition [Man82, p.15] that a fractal is a topological space with Hausdorff dimension different from its covering dimension, we see that the attractor of an iterated function system is often a fractal. This is evident in Example 1.1.10 and Example 1.1.11.

Example 1.1.15. Let (\mathbb{A}, Γ) be the iterated function system of Example 1.1.10 restricted to the Sierpinski gasket. Since each $\gamma \in \Gamma$ is a similarity with $d(\gamma_i(x), \gamma_i(y)) = \frac{1}{2}d(x, y)$ for all $x, y \in \mathbb{A}$ and $1 \leq i \leq 3$, an application of Theorem 1.1.4 implies that the Hausdorff dimension of the Sierpinski gasket is $\frac{\ln(3)}{\ln(2)}$. \triangle

1.2 | Code maps and critical Sets

For each $N \in \mathbb{N}$ we denote by $\Omega_N := \{1, 2, \dots, N\}^{\mathbb{N}}$, the full one-sided shift space on $\{1, 2, \dots, N\}$. For elements $w \in \Omega_N$ we write $w = w_1 w_2 \dots$. Let $\sigma: \Omega_N \rightarrow \Omega_N$ denote the *left-shift*

$$\sigma(w_1 w_2 w_3 \dots) = w_2 w_3 \dots \quad (1.4)$$

For each $1 \leq i \leq N$, let $\bar{\gamma}_i: \Omega_N \rightarrow \Omega_N$ denote the *inverse branch (section)* of σ by

$$\bar{\gamma}_i(w_1 w_2 \dots) = i w_1 w_2 \dots \quad (1.5)$$

and let $\bar{\Gamma} = \{\bar{\gamma}_1, \dots, \bar{\gamma}_N\}$. Define a metric $d: \Omega_N \times \Omega_N \rightarrow [0, \infty)$ by

$$d(w, v) = \begin{cases} 0 & \text{if } w = v \\ 2^{-\min\{k | w_k \neq v_k\}} & \text{otherwise.} \end{cases}$$

Then Ω_N is complete with respect to d , and each $\bar{\gamma}_i \in \bar{\Gamma}$ is a contraction. Consequently, $(\Omega_N, \bar{\Gamma})$ is a contractive iterated function system. Since Ω_N is $\bar{\Gamma}$ -invariant, Hutchinson's Theorem implies that it is the attractor of $(\Omega_N, \bar{\Gamma})$.

Definition 1.2.1. We call the contractive iterated function system $(\Omega_N, \bar{\Gamma})$ the *code space on N letters*.

Notation 1.2.2. We identify the collection of finite words over the alphabet $\{1, \dots, N\}$ with the free monoid on N generators, \mathbb{F}_N^+ . If $w \in \mathbb{F}_N^+$ then we write $w = w_1 w_2 \dots w_k$, where each $w_i \in \{1, \dots, N\}$, in which case we say that the *length* $\ell(w)$ of w is k . Denote the identity of \mathbb{F}_N^+ by \emptyset , which we think of as the empty word. Define a partial order on \mathbb{F}_N^+ by $w \leq v$ if and only if $w^{-1}v \in \mathbb{F}_N^+$, where the inverse is taken in the free group \mathbb{F}_N . We denote the least upper bound of $w, v \in \mathbb{F}_N^+$ by $w \vee v$. If w and v have no common upper bound, we write $w \vee v = \infty$.

The metric topology on Ω_N has a basis consisting of the cylinder sets

$$\mathcal{Z}(w) := \{v \in \Omega_N \mid v_i = w_i \text{ for all } 1 \leq i \leq \ell(w)\}$$

indexed by $w \in \mathbb{F}_N^+$. With this topology, Ω_N is homeomorphic to the Cantor set.

Hutchinson [Hut81] was the first to observe that there is a canonical map from a code space to the attractor of a contractive iterated function system.

Notation 1.2.3. Let $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ be an iterated function system. For each $w = w_1 \cdots w_k \in \mathbb{F}_N^+$ we write,

$$\gamma_w := \gamma_{w_1} \circ \cdots \circ \gamma_{w_k}. \quad (1.6)$$

We sometimes use the shorthand,

$$\mathbb{A}_w := \gamma_w(\mathbb{A}). \quad (1.7)$$

Theorem 1.2.4 ([Hat85, Theorem 3.2]). *Let $(X, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ be a contractive iterated function system with attractor \mathbb{A} . Then the map $\pi: \Omega_N \rightarrow \mathbb{A}$ defined by*

$$\{\pi(w)\} = \bigcap_{k \in \mathbb{N}} \mathbb{A}_{w_1 w_2 \cdots w_k} \quad (1.8)$$

for all $w = w_1 w_2 \cdots \in \Omega_N$ is a continuous surjection, and for all $1 \leq i \leq N$ we have $\gamma_i \circ \pi = \pi \circ \bar{\gamma}_i$.

Example 1.2.5. Let $\mathbb{A} = [0, 1]$ and $\Gamma = \{\gamma_0, \gamma_1\}$, where $\gamma_0(x) = \frac{x}{2}$ and $\gamma_1(x) = \frac{x+1}{2}$. Then (\mathbb{A}, Γ) is a contractive iterated function system with attractor \mathbb{A} . Write $\dot{0} = 000 \cdots$ and $\dot{1} = 111 \cdots$. Direct application of (1.8) shows that $\pi(\dot{0}) = 0$ and $\pi(\dot{1}) = 1$. Indeed, 0 is the unique fixed-point of γ_0 and 1 is the unique fixed-point of γ_1 . It follows that $\pi(1\dot{0}) = \frac{1}{2} = \pi(0\dot{1})$. Any other point $x \in [0, 1]$ with $|\pi^{-1}(x)| > 1$ is the image of $\frac{1}{2}$ under a finite sequence of maps in Γ (see Figure 1.4). Accordingly, the points 0, 1 and $\frac{1}{2}$ are of interest in relation to how the topology of $[0, 1]$ interacts with the dynamics induced by Γ .

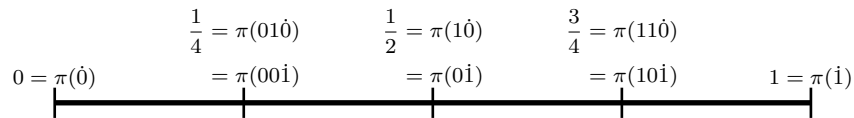


Figure 1.4: The interval $[0, 1]$ labelled using π from Example 1.2.5.

As it turns out, the map π takes a string $\omega \in \Omega_2$ to the real number represented by the binary expansion $0.\omega_1\omega_2 \cdots$. The points for which $|\pi^{-1}(x)| = 2$ correspond to those real numbers with two distinct binary representations. This example can be modified with a suitable choice of 10 contractions in such a way that π assigns a string in Ω_{10} to the decimal number it represents. \triangletriangle

Example 1.2.6. Let (X, Γ) be the iterated function system of Example 1.1.9 whose attractor \mathbb{A} is equal to the middle-thirds cantor set. Then the map $\pi: \Omega_2 \rightarrow \mathbb{A}$ is a homeomorphism from Ω_2 onto the middle-thirds cantor set. In particular $|\pi^{-1}(x)| = 1$ for all $x \in \mathbb{A}$. \triangletriangle

Remark 1.2.7. Let $([0, 1], \Gamma)$ denote the contractive iterated function system of Example 1.2.5 and let $([0, 1], \Gamma')$ denote the contractive iterated function system of Example 1.2.6. The only difference between the two pairs of contractive maps is a scaling factor. For $([0, 1], \Gamma)$, the scaling factor is $1/2$, while for $([0, 1], \Gamma')$ the scaling factor is $1/3$.

Despite this small difference, the attractor $[0, 1]$ of $([0, 1], \Gamma)$ consists of a single connected component, while the attractor of $([0, 1], \Gamma')$ is totally disconnected. This is an indication that the overlap structure of the maps plays a central role in the topology of the attractor; a fact we will revisit.

We now move away from the contractive case and back to the general setting of Definition 1.0.1.

Definition 1.2.8. We say that an iterated function system $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ admits a *code map* π if there is a continuous surjection $\pi: \Omega_N \rightarrow \mathbb{A}$ such that $\gamma_i \circ \pi = \pi \circ \bar{\gamma}_i$ for all $1 \leq i \leq N$. In this case we call $(\Omega_N, \bar{\Gamma})$ the *code space* of (\mathbb{A}, Γ) , and call elements of $\pi^{-1}(x)$ *addresses* of x .

Iterated function systems admitting code maps are called *self-similar structures* by Kigami [Kig01, Definiton 1.3.1] but are also referred to as *topological self-similar systems* (cf. [Kam00]). If $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ admits a code map π , then (1.1) is automatically satisfied since

$$\mathbb{A} = \pi(\Omega_N) = \pi\left(\bigcup_{i=1}^N \bar{\gamma}_i(\Omega_N)\right) = \bigcup_{i=1}^N \gamma_i(\mathbb{A}).$$

Moreover, Γ can be recovered since $\gamma_i(x) = \pi \circ \bar{\gamma}_i \circ \pi^{-1}(x)$ for each $x \in \mathbb{A}$. Less obviously, if (\mathbb{A}, Γ) admits a code map then metrizability of \mathbb{A} is automatic (c.f. [Kam00, Theorem 1.5]).

If $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ admits a code map $\pi: \Omega_N \rightarrow \mathbb{A}$ then the code map is unique, and satisfies

$$\{\pi(w)\} = \bigcap_{k \in \mathbb{N}} \gamma_{w_1 w_2 \dots w_k}(\mathbb{A}) \quad (1.9)$$

for all $w = w_1 w_2 \dots \in \Omega_N$ (see [Kig01, Proposition 1.3.3]).

Not every iterated function system (\mathbb{A}, Γ) admits a code map. For example, if \mathbb{A} is a space consisting of more than one point then $(\mathbb{A}, \{\text{id}_X\})$ has no code map since the system would need to satisfy (1.9). The right-hand side of (1.9) makes sense however, and a more general set-valued code map can be defined. This is the approach taken by Kieninger [Kie02, §4.2].

Given a code map we can consider its set of preimages.

Definition 1.2.9 ([Kam00, Definition 1.18]). Let (\mathbb{A}, Γ) be an iterated function system with code map $\pi: \Omega_{|\Gamma|} \rightarrow \mathbb{A}$. The collection $\{\pi^{-1}(x) \mid x \in \mathbb{A}\}$ is called the *kneading invariant* of (\mathbb{A}, Γ) .

If (\mathbb{A}, Γ) admits a code map, then \mathbb{A} can be recovered as a quotient of $\Omega_{|\Gamma|}$ using the kneading invariant (see [Kam00, §1.3]). Indeed, if we define an equivalence relation on $\Omega_{|\Gamma|}$ by $w \sim v$ if there exists $x \in \mathbb{A}$ such that $w, v \in \pi^{-1}(x)$, then $\mathbb{A} \simeq \Omega_{|\Gamma|} / \sim$.

Example 1.2.10. The contractive iterated function systems $([0, 1], \Gamma)$ from Example 1.2.5 and (C, Γ') from Example 1.2.6 have different kneading invariants. This follows from the fact that each $x \in C$ has a unique address under $([0, 1], \Gamma')$, while there are elements of $[0, 1]$ with two addresses under $([0, 1], \Gamma)$. \blacktriangle

The noted difference between the attractors of Example 1.2.5 and Example 1.2.6 is an example of a more general situation.

Theorem 1.2.11 ([Bar93, Theorem 2.2, p. 125]). *Let (X, Γ) be a contractive iterated function system with attractor \mathbb{A} . Then \mathbb{A} is totally disconnected if and only if*

$$\gamma(\mathbb{A}) \cap \gamma'(\mathbb{A}) = \emptyset$$

for all $\gamma \neq \gamma' \in \Gamma$. In this case each $x \in \mathbb{A}$ possesses a unique address: that is $|\pi^{-1}(x)| = 1$ for each $x \in \mathbb{A}$.

In the setting of contractive iterated function systems Theorem 1.2.11 shows that the sets $\gamma(\mathbb{A}) \cap \gamma'(\mathbb{A})$ for $\gamma, \gamma' \in \Gamma$ play a non-trivial role in the topological structure of the attractor \mathbb{A} . For a general injective iterated function system, we make the following definition—variations of which can be found throughout the literature (cf. [Kig01; Kam00]).

Definition 1.2.12. Let (\mathbb{A}, Γ) be an iterated function system. The *critical set* (or *set of overlap*) of (\mathbb{A}, Γ) is the set

$$C_\Gamma := \bigcup_{\gamma \neq \gamma' \in \Gamma} \gamma(\mathbb{A}) \cap \gamma'(\mathbb{A}).$$

The *post-critical set* of (\mathbb{A}, Γ) is defined to be

$$P_\Gamma := \bigcup_{w \in \mathbb{F}_N^+ \setminus \{\emptyset\}} \gamma_w^{-1}(C_\Gamma).$$

In particular, P_Γ is the collection of preimages of C_Γ . We say that (\mathbb{A}, Γ) is *post-critically finite* if P_Γ is a finite set.

Remark 1.2.13. The prefix *post* in the *post-critical set* refers to the shift map on code space. Indeed, if (\mathbb{A}, Γ) admits a code map $\pi: \Omega_N \rightarrow \mathbb{A}$, then $x \in P_\Gamma$ if and only if there exists $w \in \Omega_N$ such that $\pi(w) \in C_\Gamma$ and there is some $k \in \mathbb{N}$ such that $\pi(\sigma^k(w)) = x$.

Remark 1.2.14. Sometimes a more general notion of critical set is required for non-injective iterated function systems, which takes into account points $x \in \mathbb{A}$ for which $|\gamma^{-1}(x)| \geq 2$ (see [Kam00, Definition 1.18]). We do not use this extended notion.

Example 1.2.15. Let $(\mathbb{A}, \Gamma = \{\gamma_1, \gamma_2, \gamma_3\})$ be the iterated function system of Example 1.1.10, where \mathbb{A} is the Sierpinski gasket with vertices $(0, 0)$, $(1/2, \sqrt{3}/2)$, and $(1, 0)$. The critical set of (\mathbb{A}, Γ) consists of three points $(1/2, 0) \in \mathbb{A}_1 \cap \mathbb{A}_3$, $(1/4, \sqrt{3}/4) \in \mathbb{A}_1 \cap \mathbb{A}_2$, and $(3/4, \sqrt{3}/4) \in \mathbb{A}_2 \cap \mathbb{A}_3$. These points can be seen in blue in Figure 1.5. The post-critical set of (\mathbb{A}, Γ) consists of the three points $(0, 0)$, $(1, 0)$, and $(1/2, \sqrt{3}/2)$, which can be seen in red in Figure 1.5. Each point of P_Γ is the fixed-point of one of the maps in Γ , which is why there are no further points in the post-critical set. \triangle

Example 1.2.16 (Hata's tree-like set [Kig01, Example 1.2.9]). In this example we give an attractor for which the union defining P_Γ requires words of length 2 in \mathbb{F}_N^+ . Fix $c \in \mathbb{C}$ such that $0 < |c|, |1 - c| < 1$. Consider the contractive iterated function system on \mathbb{C} with maps $\Gamma = \{\gamma_1, \gamma_2\}$ defined by

$$\gamma_1(z) = c\bar{z} \quad \text{and} \quad \gamma_2(z) = |1 - c|^2 \bar{z} + |z|^2.$$

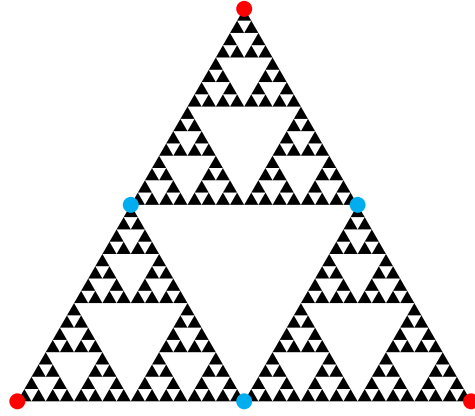


Figure 1.5: The post-critical set of Example 1.2.15 is highlighted in red. The critical set C_Γ is highlighted in blue.

Starting with the compact set $K = [0, 1] \cup c[0, 1] \subseteq \mathbb{C}$ and applying the Hutchinson operator we obtain approximations of the attractor \mathbb{A} . The attractor is known as Hata's tree-like set (see Figure 1.6).

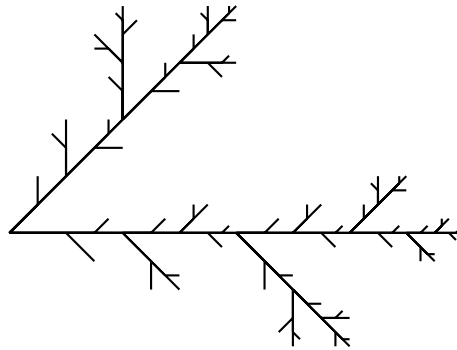


Figure 1.6: An approximation of Hata's tree-like set, for $c = \frac{1}{2}(1 + i)$.

Note that 0 is the fixed point of γ_1 , while 1 is the fixed point of γ_2 . The critical set is given by $C_\Gamma = \{|c|^2\}$. Since $|c|^2 = \gamma_1(c) = \gamma_2(0) = \gamma_{12}(1)$ it follows that $P_\Gamma = \{0, 1, c\}$.

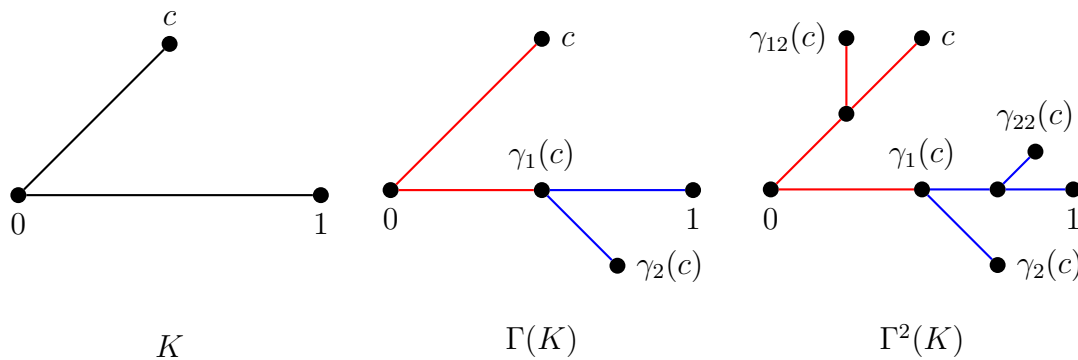


Figure 1.7: The first two iterates of the Hutchinson operator Γ applied to K . Here $c = \frac{1}{2}(1 + i)$.

Iterated function systems with code maps and finite post-critical sets provide a suitable framework for doing analysis on fractals. An introduction to the theory of differential operators on the attractors of post-critically finite iterated function systems can be found in [Kig01].

Of course, not even every contractive iterated function system has finite critical or post-critical set.

Example 1.2.17. Let $([0, 1], \Gamma = \{\gamma_1, \gamma_2\})$ be the iterated function system with maps $\gamma_1(x) = 2x/3$ and $\gamma_2(x) = 2x/3 + 1/3$. Then $C_\Gamma = [1/3, 2/3]$ while $P_\Gamma = [0, 1/2] \cup [1/2, 1] = [0, 1]$. \triangletriangle

Example 1.2.18. Let (\mathbb{A}, Γ) be the iterated function system from Example 1.1.11 with \mathbb{A} being the Sierpinski carpet. The post-critical and critical sets are shown in Figure 1.8. Both the critical set and the post-critical set are infinite, and have non-trivial topology in the sense that they are both homotopic to the circle.

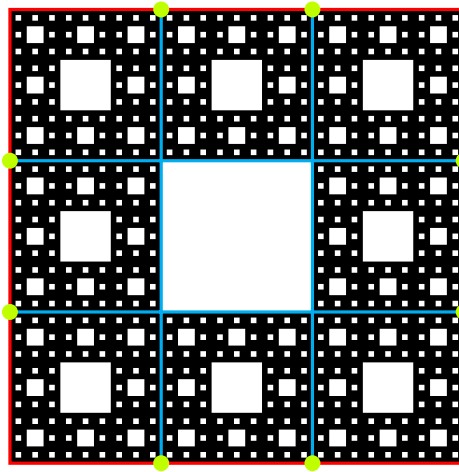


Figure 1.8: The attractor of Example 1.2.18. The critical set is highlighted blue, and the post-critical set is highlighted red. Points lying in the intersection of the two sets are green.

\triangletriangle

Definition 1.2.19. We say that an injective iterated function system (\mathbb{A}, Γ) has *fat overlap* if C_Γ has non-empty interior.

A contractive iterated function system with fat overlap cannot satisfy the open-set condition.

The relationship between the sets $\gamma(\mathbb{A}) \cap \gamma'(\mathbb{A})$ and the connectivity of \mathbb{A} was first observed by Hata [Hat85, §4] in the case of contractive iterated function systems. This analysis was further extended by Kameyama [Kam00, §1.5] to general iterated function systems admitting a code map. Part of the analysis uses what Hata and Kameyama both refer to as *chains*. Following Kameyama, a chain is an ordered tuple (w^1, w^2, \dots, w^l) of words in \mathbb{F}_N^+ satisfying $\gamma_{w^i}(\mathbb{A}) \cap \gamma_{w^{i+1}}(\mathbb{A}) \neq \emptyset$ for all $1 \leq i < l$, and $\mathcal{Z}(w^i) \cap \mathcal{Z}(w^j) = \emptyset$ for all $i \neq j$.

The following is a generalisation of Theorem 1.2.11.

Proposition 1.2.20 ([Kam00, Proposition 1.29]). *Suppose that (\mathbb{A}, Γ) is an iterated function system admitting a code map. Then x, y are contained in distinct connected components of \mathbb{A} if and only if there is no chain (w^1, w^2, \dots, w^l) with $x \in \gamma_{w^1}(\mathbb{A})$ and $y \in \gamma_{w^l}(\mathbb{A})$.*

Kameyama also proved the following remarkable result.

Theorem 1.2.21 ([Kam00, Theorem 2.4]). *Suppose that (\mathbb{A}, Γ) is an iterated function system admitting a code map. If C_Γ is finite and C_Γ contains no cluster points of P_Γ , then (\mathbb{A}, Γ) admits a metric making it a contractive iterated function system.*

Theorem 1.2.21 indicates that an iterated function system admitting a code map is not a long way from it being contractive.

Clearly the overlap structure of an iterated function system plays an important role in determining the topological structure of the attractor. The relation between connectivity of the attractor and C_Γ can be seen Proposition 1.2.20. On the other hand, examples such as the Sierpinski Gasket (Example 1.1.10) indicate that higher order topological data—such as being simply connected—is also influenced by the overlap structure.

For nice enough topological spaces, the properties of connectedness and simple connectedness are usually measured by (co)homology groups. Naturally, the following question arises.

Question 1. *Can the critical set of an iterated function system be interpreted in terms of algebraic-topological data built from the dynamical system (\mathbb{A}, Γ) ?*

The original intent, and one of the overarching aims of this thesis is to determine if it is possible to answer this question using C^* -algebras and their invariants. In particular, we would like to use the K -theory of C^* -algebras which is both a robust and computable homology theory. The procedure that we undertake—dubbed the C^* -game^a—goes as follows:

- (1) associate a C^* -algebra to the dynamical system (\mathbb{A}, Γ) ;
- (2) compute invariants of the C^* -algebra (eg. K -theory); then
- (3) infer properties of the original system (\mathbb{A}, Γ) from the invariants.

Playing the C^* -game has proved fruitful previously. For example Cuntz and Krieger [CK80] realised the Bowen-Franks group [BF77] as the K_1 -group of a C^* -algebras \mathcal{O}_A , now known as a Cuntz-Krieger algebra. The Bowen-Franks group is an invariant for flow equivalence of topological Markov chains.

For examples such as the Sierpinski Gasket (Example 1.1.10), the first Čech cohomology group of the attractor is infinitely generated. However, each of “holes” appearing the gasket are images of the central hole under the dynamics of the iterated function system. In other words, the dynamics of the iterated function system can be used to identify the holes. Hence, we would like a (co)homology theory which uses the dynamics of the iterated function system to identify the holes in this manner.

Question 2. *Is there a reasonable (co)homology theory for iterated function systems for which uses the dynamics, and is such that the (co)homology groups associated to Example 1.1.10 are finitely generated?*

Answering either of above questions is a difficult task. In Chapter 2 we generalise a previous construction by Kajiwara and Watatani to the fat overlap setting. However, the K -theory of the associated C^* -algebras only detects part of the critical set: the branched set (see Definition 2.2.10).

^aThe author would like to thank Robin Deeley for this name.

In Chapter 4 we do make some progress towards answering the above questions. We define a new C^* -algebra, called the *lacunary algebra*. Preliminary K -theory calculations indicate that this C^* -algebra is sensitive to the structure of the critical set.

We mention that a cohomology theory—called interaction cohomology—was introduced by Sumi [Sum09], which works for iterated function systems with large overlaps. The interaction cohomology groups are built using the iterated function system (\mathbb{A}, Γ) . However, for the Sierpinski Gasket (Example 1.1.10) the first interaction cohomology group agrees with Čech cohomology and is therefore infinitely generated [Sum09, Example 3.40].

Another source of inspiration for Question 2 is Putnam’s homology theory for Smale spaces [Put14]. Smale spaces are topological spaces endowed with a discrete-time dynamics which can be locally decomposed into expanding and contracting directions. As such, the theory of Smale spaces is not entirely distant from the dynamics of iterated function systems. In [Put14], Putnam constructs a homology theory for Smale spaces built from the underlying dynamics for which each homology group has finite rank [Put14, Theorem 5.1.12]. Moreover, the number of periodic points in a Smale space can be determined with a Lefschetz-type formula [Put14, Theorem 6.1.1].

Although not entirely related to iterated function systems themselves, we mention that the Sierpinski gasket has appeared in Wieler’s [Wie14, Example 3] construction a Smale space exhibiting some desirable properties.

1.3 | Morphisms and conjugacy

The categorical properties of iterated function systems have not been thoroughly explored in the literature. Having multiple functions implementing the dynamics means that there is a lot of choice when it comes to defining an appropriate notion of morphism. For our purposes, we take the following naive notion of morphisms.

Definition 1.3.1. A *morphism* from an iterated function system (\mathbb{A}, Γ) to (\mathbb{B}, Λ) is a pair (f, ψ) consisting of a continuous map $f : \mathbb{A} \rightarrow \mathbb{B}$ and a function $\psi : \Gamma \rightarrow \Lambda$ such for each $\gamma \in \Gamma$ we have

$$\psi(\gamma) \circ f = f \circ \gamma. \quad (1.10)$$

We write $(f, \psi) : (\mathbb{A}, \Gamma) \rightarrow (\mathbb{B}, \Lambda)$ to mean that (f, ψ) is a morphism from (\mathbb{A}, Γ) to (\mathbb{B}, Λ) . We mention some special types of morphism:

- (i) (f, ψ) is an *embedding* if both f and ψ are injective. In this case we say that (\mathbb{A}, Γ) is a *subsystem* of (\mathbb{B}, Λ) .
- (ii) (f, ψ) is a *semiconjugacy* or *factor map* if both f and ψ are surjective.
- (iii) (f, ψ) is an *isomorphism* or *conjugacy* if f is a homeomorphism and ψ is bijective. In this case (\mathbb{A}, Γ) is said to be *isomorphic* or *conjugate* to (\mathbb{B}, Λ) .

Remark 1.3.2. Barnsley has introduced a notion of fractal transformation [Bar06, Definition 4.15.1] which differs from our definition of morphism and relies on the code space structure of a system. Other notions of morphism between multi-function dynamical systems are explored by Kieninger [Kie02] and Dor-On [DO18].

As we will see in Corollary 1.3.8, morphisms satisfying Definition 1.3.1 tend to be rare. We now give examples of each type of morphism.

Example 1.3.3. Semiconjugacies $(\pi, \psi): (\Omega_N, \bar{\Gamma}) \rightarrow (\mathbb{A}, \Gamma)$ for which ψ is a bijection coincide with code maps (cf. Definition 1.2.8). \triangle

Example 1.3.4. Let $(\mathbb{A} = [0, 1], \Gamma = \{\gamma_1, \gamma_2\})$ be the iterated function system of Example 1.2.5, and let $(\mathbb{A}', \Gamma' = \{\gamma'_1, \gamma'_2, \gamma'_3\})$ be the iterated function system of Example 1.1.10, so that \mathbb{A}' is a Sierpinski gasket. Let $f: [0, 1] \rightarrow \mathbb{A}'$ denote the embedding of the interval into the left-hand edge of Figure 1.9 which takes 0 to the bottom left vertex, and 1 to the top vertex. Let $\psi: \Gamma \rightarrow \Gamma'$ take γ_1 to γ'_1 and γ_2 to γ'_2 . Then (f, ψ) is an embedding. \triangle

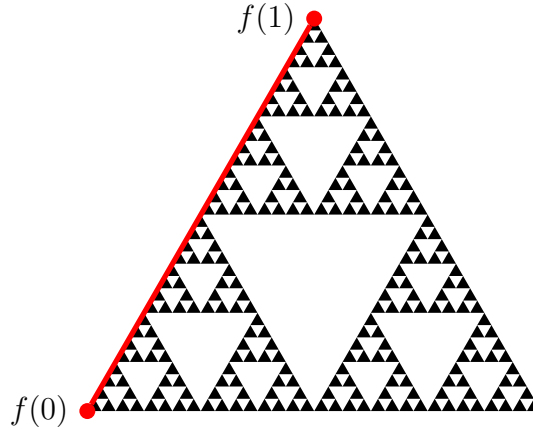


Figure 1.9: The embedding of $([0, 1], \Gamma)$ —highlighted in red—into (\mathbb{A}', Γ') of Example 1.3.4.

In practice, determining whether two iterated function systems are conjugate can be difficult. Accordingly, we would like to find good conjugacy invariants for iterated function systems. The critical set is one such invariant, and the following is an immediate consequence of its definition.

Lemma 1.3.5. *If (\mathbb{A}, Γ) and (\mathbb{B}, Λ) are conjugate iterated function systems, then their critical sets C_Γ and C_Λ are homeomorphic.*

Of course the critical set is too coarse to be a complete conjugacy invariant, but it can still be used to distinguish many systems.

Example 1.3.6. Fix $t \in (0, 1)$ and consider the contractive iterated function system on $[0, 1]$ with maps $\Gamma_t = (\gamma_1^t, \gamma_2^t)$ given by $\gamma_1^t(x) = tx$ and $\gamma_2^t(x) = tx + 1 - t$. For $t = 1/3$ this coincides with Example 1.1.9, for $t = 1/2$ this coincides with Example 1.2.5, and for $t = 2/3$ this coincides with Example 1.2.17. Let \mathbb{A}_t be the attractor of $([0, 1], \Gamma_t)$. Then \mathbb{A}_t is a Cantor set whenever $t \in (0, 1/2)$, while $\mathbb{A}_t = [0, 1]$ for $t \in [1/2, 1)$. In particular (\mathbb{A}_t, Γ_t) is not conjugate to (\mathbb{A}_s, Γ_s) whenever $t \in (0, 1/2)$ and $s \in [1/2, 1)$.

If both $s, t \in (0, 1/2)$, then (\mathbb{A}_s, Γ_s) is conjugate to (\mathbb{A}_t, Γ_t) . Indeed, both (\mathbb{A}_s, Γ_s) and (\mathbb{A}_t, Γ_t) are conjugate to the code space $(\Omega_2, \bar{\Gamma})$ via their code maps. Note that $C_{\Gamma_{1/2}} = \{1/2\}$, while for $s \in (1/2, 1)$ we have $C_{\Gamma_s} = [1 - s, s]$. Since the critical set is a conjugacy invariant $(\mathbb{A}_{1/2}, \Gamma_{1/2})$ is not conjugate to (\mathbb{A}_s, Γ_s) for $s \in (1/2, 1)$.

Lastly, we consider the case where $s, t \in (1/2, 1)$. Despite the critical sets C_{Γ_s} and C_{Γ_t} being homeomorphic, it is not immediately apparent whether (\mathbb{A}_s, Γ_s) and (\mathbb{A}_t, Γ_t) are conjugate. A homeomorphism $f: \mathbb{A}_s \rightarrow \mathbb{A}_t$ implementing a conjugacy would simultaneously solve the functional equations

$$f(sx) = tf(x) \quad \text{and} \quad f(sx + 1 - s) = tf(x) + 1 - t. \quad (1.11)$$

Finding whether solutions exist to functional equations is notoriously difficult, so we leave the existence of such an f open. In Example 1.3.12, we see that numerical evidence suggests that such an f does not exist for particular values of s and t . \triangleleft

A key observation from Example 1.3.6 is that equations (1.11) indicate a self-similarity structure to the graph $\text{Gr}(f) = \{(x, f(x)) \mid x \in [0, 1]\}$ of f . Indeed, if $(x, f(x)) \in \text{Gr}(f)$, then $(sx, tf(x))$ and $(sx + 1 - s, tf(x) + 1 - t)$ also belong to $\text{Gr}(f)$. This is due to a more general phenomenon, namely that morphisms between iterated function systems correspond to invariant sets. It is unclear to the author whether this result exists in the iterated function system literature, however similar ideas do appear throughout [Bar06, Chapter 4] in relation to fractal tops.

Proposition 1.3.7. *Let $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ and $(\mathbb{B}, \Lambda = \{\lambda_1, \dots, \lambda_N\})$ be iterated function systems. Suppose that $f: \mathbb{A} \rightarrow \mathbb{B}$ is a function such that $f \circ \gamma_i = \lambda_i \circ f$ for all $1 \leq i \leq N$. Consider the maps $g_i: \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{A} \times \mathbb{B}$ given by $g_i(x, y) = (\gamma_i(x), \lambda_i(y))$ for all $1 \leq i \leq N$. Then*

$$\text{Gr}(f) = \bigcup_{i=1}^N g_i(\text{Gr}(f)).$$

Proof. Since $f \circ \gamma_i = \lambda_i \circ f$, whenever $(x, y) \in \text{Gr}(f)$ we have $g_i(x, y) = (\gamma_i(x), \lambda_i(y)) \in \text{Gr}(f)$. Therefore,

$$\bigcup_{i=1}^N g_i(\text{Gr}(f)) \subseteq \text{Gr}(f).$$

For the reverse inclusion fix $(x, y) \in \text{Gr}(f)$. Since $\mathbb{A} = \bigcup_{i=1}^N \gamma_i(\mathbb{A})$, for each $x \in \mathbb{A}$ there exists $1 \leq i \leq N$ and $x_0 \in \mathbb{A}$ such that $x = \gamma_i(x_0)$. Let $y_0 = f(x_0)$ so that $(x_0, y_0) \in \text{Gr}(f)$. It follows that,

$$g_i(x_0, y_0) = (\gamma_i(x_0), \lambda_i(y_0)) = (x, \lambda_i \circ f(x_0)) = (x, f \circ \gamma_i(x_0)) = (x, f(x)) = (x, y),$$

so $(x, y) \in g_i(\text{Gr}(f))$. \square

Proposition 1.3.7 is particularly interesting in the contractive case.

Corollary 1.3.8. *Suppose that (\mathbb{A}, Γ) , (\mathbb{B}, Λ) , f , and $\{g_i \mid 1 \leq i \leq N\}$ satisfy the same hypotheses as Proposition 1.3.7. In addition, suppose that (\mathbb{A}, Γ) and (\mathbb{B}, Λ) are contractive, and that $f: \mathbb{A} \rightarrow \mathbb{B}$ is continuous. Then $\text{Gr}(f)$ is the unique attractor of the contractive system $\{g_i\}_{i=1}^N$ on $\mathbb{A} \times \mathbb{B}$. In particular, there is at most one continuous function $f: \mathbb{A} \rightarrow \mathbb{B}$ satisfying $f \circ \gamma_i = \lambda_i \circ f$ for all $1 \leq i \leq N$.*

Proof. It follows from Proposition 1.3.7 that $\text{Gr}(f) = \bigcup_{i=1}^N g_i(\text{Gr}(f))$. Equip $\mathbb{A} \times \mathbb{B}$ with the metric $d_\infty((x_1, y_1), (x_2, y_2)) = \max\{d_\mathbb{A}(x_1, x_2), d_\mathbb{B}(y_1, y_2)\}$. Then $\mathbb{A} \times \mathbb{B}$ is a complete metric space and each $g_i: \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{A} \times \mathbb{B}$ is a strict contraction. Since f is continuous, $\text{Gr}(f)$ is closed in $\mathbb{A} \times \mathbb{B}$, hence compact. Hutchinson's Theorem (Theorem 1.1.4) now gives the result. \square

Corollary 1.3.8 can also be restated in terms of morphisms. This gives a way to determine whether a morphism exists between two iterated function systems.

Corollary 1.3.9. *Let (\mathbb{A}, Γ) and (\mathbb{B}, Λ) be contractive iterated function systems with attractors \mathbb{A} and \mathbb{B} respectively, and suppose that $\psi: \Gamma \rightarrow \Lambda$ is a bijection. Endow $\mathbb{A} \times \mathbb{B}$ with the metric $d_\infty((x_1, y_1), (x_2, y_2)) = \max\{d_\mathbb{A}(x_1, x_2), d_\mathbb{B}(y_1, y_2)\}$ and let \mathbb{D} denote the*

attractor of the system $(\mathbb{A} \times \mathbb{B}, \{(x, y) \mapsto (\gamma(x), \psi(\gamma)(y))\}_{\gamma \in \Gamma})$. Then there is a continuous function $f: \mathbb{A} \rightarrow \mathbb{B}$ such that $(f, \psi): (\mathbb{A}, \Gamma) \rightarrow (\mathbb{B}, \Lambda)$ is a morphism if and only if \mathbb{D} is the graph of a function. Moreover, (f, ψ) is a conjugacy if and only if \mathbb{D} is the graph of a bijection.

Remark 1.3.10. In the statement of Corollary 1.3.9 we do not need to stipulate that \mathbb{D} is the graph of a *continuous* function since \mathbb{D} is already closed in $\mathbb{A} \times \mathbb{B}$.

Corollary 1.3.9 yields a proof of the well-known fact that code maps are unique.

Corollary 1.3.11 (cf. [Kig01, Proposition 1.3.3]). *Let $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ be a contractive iterated function system with attractor \mathbb{A} . Then the graph of the code map $\pi: \Omega_N \rightarrow \mathbb{A}$ is the attractor of the iterated function system $\{(x, \omega) \mapsto (\gamma_i(x), \bar{\gamma}_i(\omega))\}_{i=1}^N$ on $\mathbb{A} \times \Omega_N$. In particular, the code map is the unique continuous map from Ω_N to \mathbb{A} that satisfies $\pi \circ \bar{\gamma}_i(\omega) = \gamma_i \circ \pi$ for all $1 \leq i \leq N$.*

Armed with Corollary 1.3.8, we make a return to the conjugacy problem of Example 1.3.6.

Example 1.3.12. Fix $s, t \in (1/2, 1)$ and let $([0, 1], \Gamma_s = \{\gamma_1^s, \gamma_2^s\})$ and $([0, 1], \Gamma_t = \{\gamma_1^t, \gamma_2^t\})$ be the contractive iterated functions systems from Example 1.3.6, with attractor $[0, 1]$. For $i = 1, 2$ let $g_i^{s,t}: [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]$ be given by $g_i^{s,t}(x, y) = (\gamma_i^s(x), \gamma_i^t(y))$. Then $([0, 1] \times [0, 1], \Lambda_{s,t} := \{g_1^{s,t}, g_2^{s,t}\})$ is a contractive iterated function with respect to the Euclidean metric.

Starting with the points $(0, 0)$ and $(1, 1)$ —which are the fixed points of $g_1^{s,t}$ and $g_2^{s,t}$ —we approximate the attractor of this system using the chaos game (cf. [Bar93]). The resulting approximations for select s and t are illustrated in Figure 1.10. Each point in

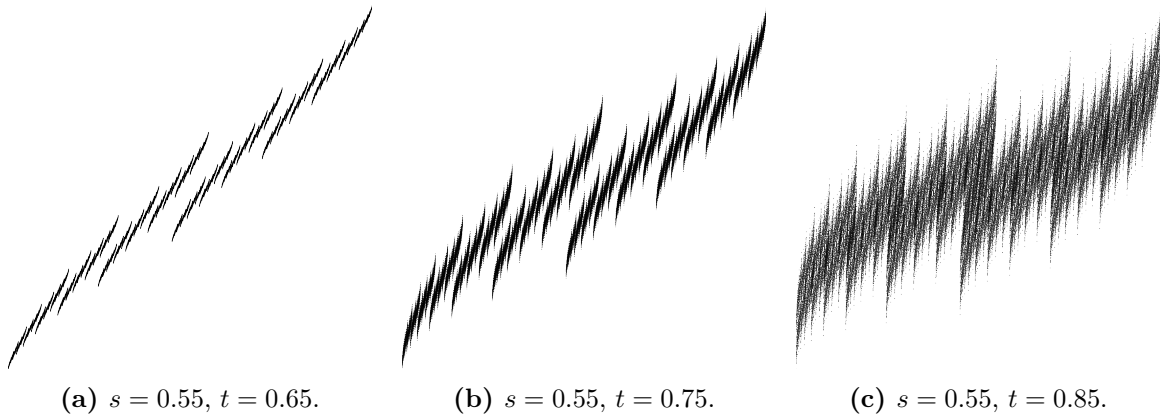


Figure 1.10: Approximations to the attractor of $([0, 1] \times [0, 1], \Lambda_{s,t})$ obtained using the chaos game.

each subplot of Figure 1.10 is obtained by applying the Hutchinson operator repeatedly to the set $\{(0, 0), (1, 1)\}$. Therefore, invariance of the attractor implies that each point in each subplot lies within its corresponding attractor.

If a homeomorphism $f: [0, 1] \rightarrow [0, 1]$ satisfying (1.11) were to exist, its graph would be the attractor $([0, 1] \times [0, 1], \Lambda_{s,t})$ by Corollary 1.3.8. For the values of s and t used in Figure 1.10 we can see that the attractor $([0, 1] \times [0, 1], \Lambda_{s,t})$ is not the graph of a continuous function. △

In light of Corollary 1.3.9, we see that morphisms between iterated functions are rare, let alone conjugacies. It is of course possible to define weaker notions of morphism and equivalence between iterated function systems. As we have previously mentioned, one of the themes of this thesis will be to associate C^* -algebras to iterated function systems and use K -theory as an invariant to distinguish between systems. This is a weak invariant, but has the advantage that it is often computable in practice.

CHAPTER 2

The Kajiwara-Watatani Approach

We now start on the journey of using C^* -algebras to examine iterated function systems. In the introduction it was mentioned that operator theoretic techniques have been previously used to study both iterated function systems and fractals in general. We take the approach of modelling iterated function systems using C^* -correspondences and Cuntz-Pimsner algebras, a summary of which can be found in Appendix A. A C^* -correspondence can be thought of a generalised morphism between C^* -algebras. If the coefficient algebras are commutative, then a C^* -correspondence could be considered “dual to a generalised morphism” of the underlying space. In this sense, C^* -correspondences provide a natural setting for studying multi-function dynamics.

The first attempt to study iterated function systems with Cuntz-Pimsner algebras was by Pinzari-Watatani-Yonetani [PWY00]. We review their algebra in Section 2.1. One issue with Pinzari-Watatani-Yonetani’s approach is that for contractive iterated function systems, the resulting Cuntz-Pimsner algebra is only sensitive to number of maps.

Pinzari-Watatani-Yonetani’s approach was later improved upon by Kajiwara and Watatani [KW06]. Their key observation was that the dynamics of an iterated function system (\mathbb{A}, Γ) is encoded entirely within its graph. Using the graph, they constructed a C^* -correspondence E_Γ and its associated Cuntz-Pimsner algebra, which we denote $C^*(\mathbb{A}, \Gamma)$. This construction is based on a correspondence they built for encoding rational functions on the Riemann sphere [KW05].

Many of Kajiwara and Watatani’s results were only proved in the setting where (\mathbb{A}, Γ) is contractive and satisfies the open-set condition. One the main outcomes in this chapter is the extension of some of Kajiwara and Watatani’s main results to iterated function systems in the general form of Definition 1.0.1. In particular, we do not require contractible systems, nor do we require the open-set condition hypothesis. The key to this is the identification of the *branched set* (see Definition 2.2.10).

Kajiwara and Watatani had already observed the dependence of their algebra on the branched structure of the underlying iterated function system (see [IKW07; KW14; KW16; KW17]). This dependence carries through to the general setting, and we highlight this with some K -theory computations in Section 2.4. A version of Kajiwara and Watatani’s algebras for Mauldin-Williams graphs has also been considered by Ionescu and Watatani [IW08].

We pay special attention to invertible iterated function systems, which were also considered by Kajiwara and Watatani. In particular, we use Kieninger’s [Kie02] idea of an inverse lifted system to show that $C^*(\mathbb{A}, \Gamma)$ always embeds in the Kajiwara-Watatani algebra of an invertible system. Moreover, we use the machinery introduced by Kwaśniewski [Kwa17] to show that the Kajiwara-Watatani algebra of every invertible system is isomor-

phic to an Exel crossed product [Exe03], generalising a result of de Castro [dCa09]. We finish this chapter by constructing a frame for Kajiwara and Watatani's correspondence within a restricted setting.

2.1 | An initial construction

Let (\mathbb{A}, Γ) be an iterated function system. In this section we describe a simple C^* -correspondence which encodes the dynamics of (\mathbb{A}, Γ) . This particular construction was first considered in the contractive case by Pinzari-Watatani-Yonetani [PWY00, §4], however we elaborate on the details of its construction, since it provides the basis for the constructions considered in Section 2.2 and Chapter 4. Relevant background and notation pertaining to Hilbert C^* -modules, C^* -correspondences, and Cuntz-Pimsner algebras can be found in Appendix A.

Construction: X_Γ To begin, let $A = C(\mathbb{A})$ and $X_\Gamma = C(\mathbb{A} \times \Gamma)$. Define a right action of $a \in A$ on $\xi \in X_\Gamma$ by

$$(\xi \cdot a)(x, \gamma) = \xi(x, \gamma)a(x)$$

and a right A -valued inner product by

$$(\xi \mid \eta)_A(x) = \sum_{\gamma \in \Gamma} \overline{\xi(x, \gamma)} \eta(x, \gamma).$$

Then X_Γ is a right Hilbert A -module, which is isomorphic to the direct sum $A_A^{|\Gamma|}$. Define a left action of $a \in A$ on $\xi \in X_\Gamma$ by,

$$(a \cdot \xi)(x, \gamma) = a(\gamma(x))\xi(x, \gamma).$$

This action is adjointable and defines a $*$ -homomorphism $\phi: A \rightarrow \text{End}_A(X_\Gamma)$. Thus, (ϕ, X_Γ) is a correspondence over A .

It is straightforward to verify that ϕ is injective and non-degenerate, and that the inner product on X_Γ is full. Moreover, since X_Γ is isomorphic as a Hilbert A -module to the finitely generated module $A_A^{|\Gamma|}$, we see that $\text{End}_A(X_\Gamma) \cong \text{End}_A^0(X_\Gamma) \cong M_{|\Gamma|}(A)$. Let $\Gamma = \{\gamma_1, \dots, \gamma_N\}$. The functions $e_1, \dots, e_N \in X_\Gamma$ given by

$$e_i(x, \gamma) = \begin{cases} 1 & \text{if } \gamma = \gamma_i \\ 0 & \text{otherwise,} \end{cases} \quad (2.1)$$

constitute a frame $(e_i)_{i=1}^N$ for X_Γ (see Definition A.1.6). △

We associate to (ϕ, X_Γ) the Cuntz-Pimsner algebra \mathcal{O}_{X_Γ} . Despite the fact that the C^* -correspondence X_Γ depends heavily on the underlying iterated function system (\mathbb{A}, Γ) , we will see in Proposition 2.1.4 that \mathcal{O}_{X_Γ} is always isomorphic to the Cuntz algebra $\mathcal{O}_{|\Gamma|}$ when the system admits a code map. In particular, the Cuntz-Pimsner construction is only sensitive to the number of maps forming Γ . Throughout this thesis we investigate a number of approaches to construct a Cuntz-Pimsner algebra which is topologically more

sensitive, but for now we continue with the correspondence X_Γ .

Before proceeding, we recall that the Cuntz algebra \mathcal{O}_N is the universal C^* -algebra generated by N isometries with mutually orthogonal ranges [Cun77]. That is,

$$\mathcal{O}_N := C^*\left(\left\{S_i \mid 1 \leq i \leq N, S_i^* S_i = 1, \text{ and } \sum_{i=1}^N S_i S_i^* = 1\right\}\right).$$

For each $w = w_1 \cdots w_k \in \mathbb{F}_N^+$ we write $S_w = S_{w_1} \cdots S_{w_k}$. A straightforward computation shows that for all $w, v \in \mathbb{F}_N^+$ we have $S_w S_w^* S_v S_v^* = S_{w \vee v} S_{w \vee v}^*$ (with the convention that $S_\infty = 0$). It follows that

$$\mathcal{D}_N := \overline{\text{span}}\{S_w S_w^* \mid w \in \mathbb{F}_N^+\}$$

defines a commutative C^* -subalgebra of \mathcal{O}_N , often called the *diagonal*. The diagonal is a maximal Abelian subalgebra, and moreover it is Cartan [Ren08, §6.3]. The spectrum of \mathcal{D}_N is homeomorphic to the Cantor space Ω_N [CK80, Proposition 2.5]. Accordingly, \mathcal{D}_N is isomorphic to $C(\Omega_N)$.

Recall that $C(\Omega_N)$ is densely spanned by the characteristic functions $\chi_{Z(w)}$, indexed by $w \in \mathbb{F}_N^+$. Letting $\chi_{Z(\infty)}$ be the zero function, we have $\chi_{Z(w)} \chi_{Z(v)} = \chi_{Z(w \vee v)}$ for all $w, v \in \mathbb{F}_N^+$. Under the identification of \mathcal{D}_N with $C(\Omega_N)$ the element $S_w S_w^*$ is sent to the characteristic function $\chi_{Z(w)}$.

It is difficult to state the importance of Cuntz algebra in the theory of operator algebras, with the seminal paper of Cuntz [Cun77] being amongst the most cited papers in the field. An early extension of Cuntz algebras was to Cuntz-Krieger algebras [CK80], which were built to encode the dynamics of shifts of finite type within a noncommutative framework. In this setting, the Cuntz algebra \mathcal{O}_N encodes the dynamics of the full shift Ω_N . Indeed, if $\sigma^*: C(\Omega_N) \rightarrow C(\Omega_N)$ is dual to the shift (1.4) and $\alpha: C(\Omega_N) \rightarrow \mathcal{D}_N$ is the isomorphism described above then—by first checking on characteristic functions—it is straightforward to show that

$$\alpha(\sigma^*(f)) = \sum_{i=1}^N S_i \alpha(f) S_i^* \quad (2.2)$$

for all $f \in C(\Omega)$. On the other hand, if $(\Omega_N, \bar{\Gamma} = \{\bar{\gamma}_1, \dots, \bar{\gamma}_N\})$ denotes the code space, then

$$\alpha(f \circ \bar{\gamma}_i) = S_i^* \alpha(f) S_i. \quad (2.3)$$

Accordingly, \mathcal{O}_N is an ideal candidate for a noncommutative model of the iterated function system $(\Omega_N, \bar{\Gamma})$.

There are many distinct ways to construct the Cuntz algebra (as Proposition 2.1.4 will highlight). In terms of Cuntz-Pimsner models and in light of (2.3), the following is perhaps the most dynamically natural, although it differs from the example given by Pimsner [Pim97, Example (2)]. The construction is well-known, but we include a proof since the ideas are central to Chapter 4.

Proposition 2.1.1. *Let $(\Omega_N, \bar{\Gamma} = \{\bar{\gamma}_1, \dots, \bar{\gamma}_N\})$ be the code space on N letters and let $(e_i)_{i=1}^N$ be the frame for $X_{\bar{\Gamma}}$ given by (2.1). Then there is a Cuntz-Pimsner covariant representation $(\alpha, \psi): (\phi, X_{\bar{\Gamma}}) \rightarrow \mathcal{O}_N$ satisfying*

$$\alpha(\chi_{Z(w)}) = S_w S_w^* \quad \text{and} \quad \psi(e_i) = S_i. \quad (2.4)$$

Moreover, the induced map $\alpha \times \psi: \mathcal{O}_{X_{\overline{F}}} \rightarrow \mathcal{O}_N$ (see Definition A.3.4) is an isomorphism.

Proof. Let α denote the isomorphism $\alpha: C(\Omega) \rightarrow \mathcal{D}_N$ satisfying $\alpha(\chi_{\mathcal{Z}(w)}) = S_w S_w^*$ composed with the inclusion of \mathcal{D}_N into \mathcal{O}_N . Define $\psi: X_{\overline{F}} \rightarrow \mathcal{O}_N$ on elements of the form $e_i \cdot \chi_{\mathcal{Z}(w)}$ by $\psi(e_i \cdot \chi_{\mathcal{Z}(w)}) = S_i S_v S_v^*$. Then,

$$\begin{aligned} \psi(e_i \cdot \chi_{\mathcal{Z}(w)})^* \psi(e_j \cdot \chi_{\mathcal{Z}(v)}) &= S_w S_w^* S_i^* S_j S_v S_v^* \\ &= \delta_{i,j} S_{w \vee v} S_{w \vee v}^* \\ &= \alpha(\chi_{\mathcal{Z}(w \vee v)}) \\ &= \alpha((e_i \cdot \chi_{\mathcal{Z}(w)} \mid e_j \cdot \chi_{\mathcal{Z}(v)})_A). \end{aligned}$$

It follows that ψ extends to an isometric linear map $\psi: X_{\overline{F}} \rightarrow \mathcal{O}_N$ satisfying $\psi^*(\xi)\psi(\xi) = \pi((\xi \mid \xi)_A)$ for all $\xi \in X_{\overline{F}}$. We also have

$$\psi(e_i \cdot \chi_{\mathcal{Z}(v)}) \alpha(\chi_{\mathcal{Z}(w)}) = S_i S_v S_v^* S_w S_w^* = S_{i(v \vee w)} S_{v \vee w}^* = \psi(e_j \cdot (\chi_{\mathcal{Z}(v)} \chi_{\mathcal{Z}(w)})).$$

Extending by continuity we see that $\psi(\xi \cdot a) = \psi(\xi) \alpha(a)$ for all $\xi \in X_{\overline{F}}$ and $a \in C(\Omega_N)$. For the left action first observe that

$$\phi(\chi_{\mathcal{Z}(w)}) e_i(v, \overline{\gamma}_j) = \delta_{i,j} \chi_{\mathcal{Z}(w)}(iv) = \begin{cases} \delta_{i,j} & \text{if } i \leq w, \text{ and } v \in \mathcal{Z}(i^{-1}w); \\ 0 & \text{if } iv \notin \mathcal{Z}(w). \end{cases}$$

It then follows that

$$\phi(\chi_{\mathcal{Z}(w)}) e_i = \begin{cases} e_i \cdot \chi_{\mathcal{Z}(i^{-1}w)} & \text{if } i \leq w; \\ 0 & \text{otherwise.} \end{cases}$$

In \mathcal{O}_N we compute:

$$\begin{aligned} \alpha(\chi_{\mathcal{Z}(w)}) \psi(e_i \cdot \chi_{\mathcal{Z}(v)}) &= S_w S_w^* S_i S_v S_v^* \\ &= \begin{cases} S_i S_{i^{-1}w}^* S_{i^{-1}w}^* S_v S_v^* & \text{if } i \leq w; \\ 0 & \text{otherwise;} \end{cases} \\ &= \begin{cases} \psi(e_i \cdot (\chi_{\mathcal{Z}(i^{-1}w)} \chi_{\mathcal{Z}(v)})) & \text{if } i \leq w; \\ 0 & \text{otherwise;} \end{cases} \\ &= \psi(\phi(\chi_{\mathcal{Z}(w)}) e_i \cdot \chi_{\mathcal{Z}(v)}). \end{aligned}$$

Extending by continuity shows that $\psi(\phi(a)\xi) = \alpha(a)\psi(\xi)$ for all $\xi \in X_{\overline{F}}$ and $a \in C(\Omega_N)$.

For Cuntz-Pimsner covariance, observe that for all $a \in C(\Omega_N)$ we have $\phi(a) = \sum_{i=1}^N \Theta_{\phi(a)e_i, e_i}$. It follows from the computation

$$\psi^{(1)}\left(\sum_{i=1}^N \Theta_{\phi(\chi_{\mathcal{Z}(w)})e_i, e_i}\right) = \sum_{i=1}^N \psi(\phi(\chi_{\mathcal{Z}(w)})e_i) \psi(e_i)^* = \sum_{i=1}^N S_w S_w^* S_i S_i^* = S_w S_w^* = \alpha(\chi_{\mathcal{Z}(w)}),$$

that $\psi^{(1)} \circ \phi(a) = \alpha(a)$ for all $a \in C(\Omega_N)$. Consequently, (α, ψ) is a Cuntz-Pimsner covariant representation of $(\phi, X_{\overline{\Gamma}})$ in \mathcal{O}_N .

Finally, let $\alpha \times \psi: \mathcal{O}_{X_{\overline{\Gamma}}} \rightarrow \mathcal{O}_N$ be the $*$ -homomorphism induced by the universal property of $\mathcal{O}_{X_{\overline{\Gamma}}}$. Since $S_i = \psi(e_i)$ is in the range of ψ , $\alpha \times \psi$ is onto. On the other hand, since $(\alpha \times \psi)(i_E(e_i)) = S_i$, the $*$ -homomorphism $\alpha \times \psi$ preserves the gauge action. It now follows from the Gauge-Invariant Uniqueness Theorem (Theorem A.3.11) that $\alpha \times \psi$ is an isomorphism. \square

The construction of the correspondence (ϕ, X_{Γ}) is functorial with respect to semiconjugacies between iterated function systems carrying the same number of maps.

Lemma 2.1.2. *Let $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ and $(\mathbb{A}', \Gamma' = \{\gamma'_1, \dots, \gamma'_N\})$ be iterated function systems with the same number of maps. Denote the associated C^* -correspondences by (ϕ, X_{Γ}) and $(\phi', X_{\Gamma'})$, respectively. Suppose that $(f, \psi): (\mathbb{A}, \Gamma) \rightarrow (\mathbb{A}', \Gamma')$ is a semiconjugacy and define $\beta: X_{\Gamma'} \rightarrow X_{\Gamma}$ by $\beta(\xi)(x, \gamma) = \xi(f(x), \psi(\gamma))$. Then (f^*, β) defines an injective covariant morphism of C^* -correspondences from $(\phi', X_{\Gamma'})$ to (ϕ, X_{Γ}) . In particular, there is an injective $*$ -homomorphism $f^* \times \beta: \mathcal{O}_{X_{\Gamma'}} \rightarrow \mathcal{O}_{X_{\Gamma}}$.*

Proof. Since $f: \mathbb{A} \rightarrow \mathbb{A}'$ is surjective, $f^*: C(\mathbb{A}) \rightarrow C(\mathbb{A}')$ is injective. For each $\xi, \eta \in X_{\Gamma}$ we have,

$$\begin{aligned} (\beta(\xi) \mid \beta(\eta))_{C(\mathbb{A}')} (x) &= \sum_{\gamma' \in \Gamma'} \overline{\xi(f(x), \psi(\gamma'))} \eta(f(x), \psi(\gamma')) \\ &= \sum_{\gamma \in \Gamma} \overline{\xi(f(x), \gamma)} \eta(f(x), \gamma) \\ &= f^*((\xi \mid \eta)_{C(\mathbb{A})})(x). \end{aligned}$$

The right actions are clearly intertwined by (f^*, β) . For the left action we use the morphism property of (f, ψ) to see that for $a \in C(\mathbb{A})$ and $\xi \in X_{\Gamma}$ we have,

$$\begin{aligned} \phi(f^*(a))\beta(\xi)(x, \gamma) &= f^*(a)(\gamma(x)) \xi(f(x), \psi(\gamma)) \\ &= a(\psi(\gamma) \circ f(a)) \xi(f(x), \psi(\gamma)) \\ &= \beta(\phi'(a)\xi)(x, \gamma). \end{aligned}$$

For covariance, let $a \in C(\mathbb{A})$ and let $(e'_i)_{i=1}^N$ denote the frame for X_{Γ} given by (2.1). Then for each $\xi \in X_{\Gamma'}$ we have

$$\sum_{i=1}^N \Theta_{\beta(e'_i), \beta(e'_i)} \xi(x, \gamma_j) = \sum_{i,k=1}^N e'_i(f(x), \psi(\gamma_j)) \overline{e'_i(f(x), \psi(\gamma_k))} \xi(x, \gamma_k) = \xi(x, \gamma_j),$$

since the only non-zero term of the double sum occurs when $\psi(\gamma_j) = \psi(\gamma_k) = i$. It now follows from Lemma A.3.16 that (f^*, β) is covariant. The result now follows from Lemma A.3.14. \square

Remark 2.1.3. The hypothesis $|\Gamma| = |\Gamma'|$ was essential in the proof of Lemma 2.1.2. Without this hypothesis β would not preserve inner products.

We finish this section by proving that for every iterated function system (\mathbb{A}, Γ) admitting a code map, the algebra \mathcal{O}_{X_Γ} is isomorphic to the Cuntz algebra $\mathcal{O}_{|\Gamma|}$. Moreover, the isomorphism is induced by a Cuntz-Pimsner covariant morphism of C^* -correspondences.

Proposition 2.1.4. *Let $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ be an iterated function system with code map $\pi: \Omega_N \rightarrow \mathbb{A}$. Let $(i_{C(\mathbb{A})}, i_{X_\Gamma}): X_\Gamma \rightarrow \mathcal{O}_{X_\Gamma}$ and $(i_{C(\Omega_N)}, i_{X_\Gamma}): X_\Gamma \rightarrow \mathcal{O}_{X_\Gamma} \cong \mathcal{O}_N$ be the associated universal representations. Let α be given by (2.4). Then there is an isomorphism $\Phi: \mathcal{O}_{X_\Gamma} \rightarrow \mathcal{O}_N$ satisfying $\Phi \circ i_{C(\mathbb{A})}(a) = \alpha \circ \pi^*(a)$ for all $a \in C(\mathbb{A})$, and $\Phi \circ i_{X_\Gamma}(e_i) = S_i$ for all $1 \leq i \leq N$.*

Proof. Since a code map defines a semiconjugacy $(\pi, \bar{\gamma}_i \mapsto \gamma_i)$, it follows from Proposition 2.1.1 and Lemma 2.1.2 that there is an injective $*$ -homomorphism $\Phi: \mathcal{O}_{X_\Gamma} \rightarrow \mathcal{O}_N$ satisfying $\Phi \circ i_{C(\mathbb{A})}(a) = \alpha \circ \pi^*(a)$ for all $a \in C(\mathbb{A})$, and $\Phi \circ i_{X_\Gamma}(e_i) = S_i$ for all $1 \leq i \leq N$. Since the isometries S_i generate \mathcal{O}_N , we see that Φ is surjective. \square

A version of Proposition 2.1.4 can be found in [PWY00, Proposition 4.7] which applies to contractive systems (in fact their result applies to contractive maps on noncommutative metric spaces). A noteworthy advantage of our formulation however, is that the covariant morphism which induces the isomorphism is built directly from the code map. Indeed, $\Phi: \mathcal{O}_{X_\Gamma} \rightarrow \mathcal{O}_N$ could be deemed a noncommutative code map since for all $a \in C(\mathbb{A})$ we have,

$$\Phi(i_{C(\mathbb{A})}(a \circ \gamma_i)) = \Phi(i_X(e_i)^* i_{C(\mathbb{A})}(a) i_X(e_i)) = S_i^* \alpha(\pi^*(a)) S_i = \alpha(\pi^*(a) \circ \bar{\gamma}_i). \quad (2.5)$$

Remark 2.1.5. Given an iterated function system (\mathbb{A}, Γ) , there is an associated action of the free monoid \mathbb{F}_N^+ on the C^* -algebra $C(\mathbb{A})$. Indeed, for each $w \in \mathbb{F}_N^+$ define $\kappa_w: C(\mathbb{A}) \rightarrow C(\mathbb{A})$ by $\kappa_w(f)(x) = f(\gamma_w(x))$ for $f \in C(\mathbb{A})$. Then $\kappa: w \mapsto \kappa_w$ is an action of \mathbb{F}_N^+ on $C(\mathbb{A})$. If (\mathbb{A}, Γ) is injective, then the κ_w is surjective for all $w \in \mathbb{F}_N^+$.

There are a number of ways to construct C^* -algebras from semigroup actions, including multiple notions of semigroup crossed-product [LR96; Exe08; KL09]. In the author's experience it is frequently the case that such crossed-products once again reproduce \mathcal{O}_N for the action κ . This usually stems from the identity $S_w^*(\alpha \circ \pi^*(a)) S_w = (\alpha \circ \pi^*)(\kappa_w(a))$ in \mathcal{O}_N . In particular, the semigroup action κ on $C(\mathbb{A})$ can be encoded using conjugation by the generating isometries of \mathcal{O}_N . We do not make use of semigroup crossed-products in this thesis. We have also not found any advantages in using Cuntz-Nica-Pimsner algebras of product systems [SY10] in the context of iterated function systems.

2.2 | The Kajiwara-Watatani algebra

In this section we outline the construction of a C^* -algebra $C^*(\mathbb{A}, \Gamma)$ associated to a (not necessarily contractive) iterated function system (\mathbb{A}, Γ) . The algebra $C^*(\mathbb{A}, \Gamma)$ is often distinct from the Cuntz-Pimsner algebra \mathcal{O}_{X_Γ} considered in the previous section, and was first considered by Kajiwara and Watatani [KW06] in the case where (\mathbb{A}, Γ) is contractive. However, the same construction remains valid in the absence of the contractive hypothesis and of the assumption that (\mathbb{A}, Γ) is injective. With the generality we work in, we also remove the open-set condition hypothesis from some of Kajiwara and Watatani's results.

To begin, for each $\gamma \in \Gamma$ we define the *graph* of γ to be the set

$$\text{Gr}(\gamma) := \{(\gamma(y), y) \mid y \in \mathbb{A}\} \subseteq \mathbb{A} \times \mathbb{A},$$

which we equip with the subspace topology of the product topology on $\mathbb{A} \times \mathbb{A}$. Since γ is continuous, $\text{Gr}(\gamma)$ is closed.

Remark 2.2.1. The coordinates of $\text{Gr}(\gamma)$ are reversed when compared to the “usual” convention of graphs (which we used in Section 1.3) and should strictly speaking be called the *cograph* of γ . This convention is taken so the composition of the dynamics aligns itself covariantly with the composition of operators when we come to construct a C^* -algebra.

Definition 2.2.2. Let (\mathbb{A}, Γ) be an iterated function system. The *graph* of Γ is the set,

$$\text{Gr}(\Gamma) := \bigcup_{\gamma \in \Gamma} \text{Gr}(\gamma) = \{(x, y) \in \mathbb{A} \times \mathbb{A} \mid x = \gamma(y) \text{ for some } \gamma \in \Gamma\}. \quad (2.6)$$

Considering Γ to be the Hutchinson operator $\Gamma: \mathbb{H}(\mathbb{A}) \rightarrow \mathbb{H}(\mathbb{A})$ defined by

$$\Gamma(S) = \bigcup_{\gamma \in \Gamma} \gamma(S),$$

Γ can be considered a multivalued function on \mathbb{A} , and $\text{Gr}(\Gamma)$ agrees with the graph of Γ considered as a multivalued function (cf. [HP97, Definition 2.10]).

Example 2.2.3. Let $\mathbb{A} = [0, 1]$ and define the following maps on \mathbb{A} :

$$\gamma_1(x) = \frac{x}{2}, \quad \gamma_2(x) = \frac{1+x}{2}, \quad \text{and} \quad \gamma'_2(x) = 1 - \frac{x}{2}.$$

Then $(\mathbb{A}, \Gamma = \{\gamma_1, \gamma_2\})$ and $(\mathbb{A}, \Gamma' = \{\gamma_1, \gamma'_2\})$ both define contractive iterated function systems on \mathbb{A} . Note that (\mathbb{A}, Γ) is the same system considered in Example 1.2.5, while Γ' consists of the inverse branches of the tent map $f: [0, 1] \rightarrow [0, 1]$,

$$f(x) = \begin{cases} 2x & \text{if } x \leq 1/2; \\ 2(1-x) & \text{if } x > 1/2. \end{cases}$$

Although (\mathbb{A}, Γ) and (\mathbb{A}, Γ') have the same attractor, $\text{Gr}(\Gamma)$ and $\text{Gr}(\Gamma')$ are not homeomorphic. This can be seen in Figure 2.1. The topology of $\text{Gr}(\Gamma)$ is a central theme for the remainder of this chapter.

△△

Definition 2.2.4. Let $p_1: \text{Gr}(\Gamma) \rightarrow \mathbb{A}$ denote the projection $p_1(x, y) = x$ and let $p_2: \text{Gr}(\Gamma) \rightarrow \mathbb{A}$ denote the projection $p_2(x, y) = y$.

The projections p_1 and p_2 play an important role in the sequel, and so we pause to look at their properties.

Lemma 2.2.5. *The projection $p_2: \text{Gr}(\Gamma) \rightarrow \mathbb{A}$ is an open map.*

Proof. For each $\gamma \in \Gamma$, let $p_2^\gamma: \text{Gr}(\gamma) \rightarrow \mathbb{A}$ denote the projection onto the second factor of $\text{Gr}(\gamma)$. Since p_2^γ is a continuous bijection from the compact space $\text{Gr}(\gamma)$ to the Hausdorff space \mathbb{A} , it follows that $p_2^\gamma: \text{Gr}(\gamma) \rightarrow \mathbb{A}$ is a homeomorphism. Fix $U \subseteq \text{Gr}(\Gamma)$ open in the subspace topology on $\text{Gr}(\Gamma)$. Then $U = \bigcup_{\gamma \in \Gamma} (U \cap \text{Gr}(\gamma))$ and $U \cap \text{Gr}(\gamma)$ is open in $\text{Gr}(\gamma)$ for each $\gamma \in \Gamma$. Consequently, $p_2(U) = \bigcup_{\gamma \in \Gamma} p_2^\gamma(U \cap \text{Gr}(\gamma))$ is open in \mathbb{A} . \square

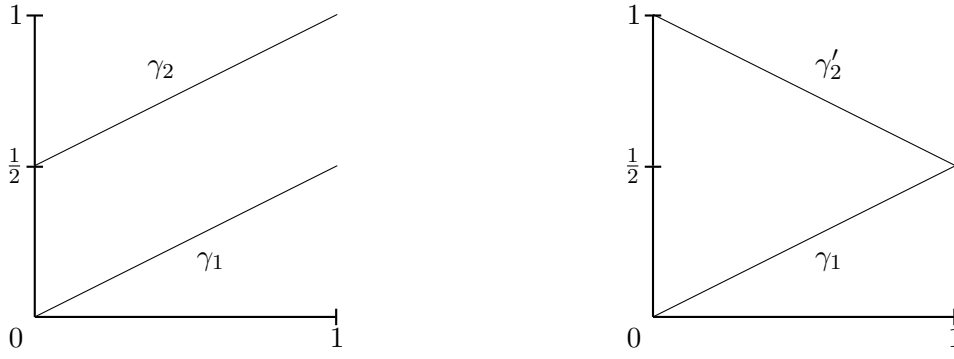


Figure 2.1: The graphs $\text{Gr}(\Gamma)$ (left) and $\text{Gr}(\Gamma')$ (right) from Example 2.2.3 viewed as subsets of $[0, 1] \times [0, 1]$. The vertical axis represents the first coordinate of $\text{Gr}(\Gamma)$.

On the other hand, we can see from the graph on the left of Figure 2.1 that the map $p_1: \text{Gr}(\Gamma) \rightarrow \mathbb{A}$ is not open in general, a fact that is further explored in Section 4.4. We recall the following.

Definition 2.2.6. A function $f: X \rightarrow Y$ between Hausdorff spaces is said to be *locally injective at* $x \in X$ if there is an open neighbourhood U of x such that f is injective when restricted to U . We say that f is *locally injective* if it is locally injective at each $x \in X$.

Another observation from the graph on the right of Figure 2.1, is that p_2 is typically not locally injective. In fact [PT11, Theorem 2.9] implies that p_2 is a branched covering in the sense of [PT11, Definition 2.4].

Construction: E_Γ Using the graph $\text{Gr}(\Gamma)$, we now construct a C^* -correspondence for (\mathbb{A}, Γ) . Let $A = C(\mathbb{A})$ and let $E_\Gamma = C(\text{Gr}(\Gamma))$. We make E_Γ into a right Hilbert A -module by defining a right action of A on E_Γ by

$$(\xi \cdot a)(x, y) = \xi(x, y) a(y)$$

for $\xi \in E_\Gamma$ and $a \in A$, and a right A -linear inner product on E_Γ by

$$(\xi \mid \eta)_A(y) := \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{\xi(\gamma(y), y)} \eta(\gamma(y), y)$$

for all $\xi, \eta \in E_\Gamma$. It is straightforward to check that E_Γ is a right inner product A -module. We take the norm on E_Γ given by $\|\xi\| := \|(\xi \mid \xi)_A\|_\infty^{1/2}$ for each $\xi \in E_\Gamma$. The normalisation $\frac{1}{|\Gamma|}$ is not strictly necessary, however it simplifies later computations.

Notation 2.2.7. For $y \in \mathbb{A}$ we define $\Gamma y = \{\gamma(y) \mid \gamma \in \Gamma\}$.

Lemma 2.2.8 ([KW06, Proposition 2.1]). *The norm on E_Γ is equivalent to the uniform norm on $C(\text{Gr}(\Gamma))$. In particular, for each $\xi \in E_\Gamma$ we have,*

$$\frac{1}{\sqrt{|\Gamma|}} \|\xi\|_\infty \leq \|\xi\| \leq \|\xi\|_\infty. \quad (2.7)$$

Proof. For each $\xi \in E_\Gamma$ we compute,

$$\begin{aligned} \frac{1}{|\Gamma|} \sup_{(x,y) \in \text{Gr}(\Gamma)} |\xi(x,y)|^2 &= \frac{1}{|\Gamma|} \sup_{y \in \mathbb{A}} \sup_{x \in \Gamma_y} |\xi(x,y)|^2 \\ &\leq \frac{1}{|\Gamma|} \sup_{y \in \mathbb{A}} \sum_{\gamma \in \Gamma} |\xi(\gamma(y), y)|^2 \\ &\leq \sup_{(x,y) \in \text{Gr}(\Gamma)} |\xi(x,y)|^2. \end{aligned}$$

Since $\|\xi\|^2 = \sup_{y \in \mathbb{A}} \sum_{\gamma \in \Gamma} |\xi(\gamma(y), y)|^2$, the result follows. \square

It follows from Lemma 2.2.8 that E_Γ is complete in norm. Hence, E_Γ is a right Hilbert A -module. The module E_Γ is full since $a = (1_{E_\Gamma} \mid 1_{E_\Gamma} \cdot a)_A$ for all $a \in A$, where $1_{E_\Gamma}(x, y) = 1$ for all $(x, y) \in \text{Gr}(\Gamma)$. Since $\text{Gr}(\Gamma)$ and \mathbb{A} are second-countable, E_Γ is countably generated.

Define a left action $\phi: A \rightarrow \text{End}_A(E_\Gamma)$ by

$$(\phi(a)\xi)(x, y) = a(x)\xi(x, y),$$

for $a \in A$ and $\xi \in E_\Gamma$. It is straightforward to check that $\phi(a)$ is adjointable with adjoint $\phi(a^*)$. Therefore, (ϕ, E_Γ) is an A - A -correspondence. As observed in [KW06, Proposition 2.1] the left action ϕ is unital by definition, and injective since $\mathbb{A} = \bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A})$. \blacktriangle

Definition 2.2.9. Let (\mathbb{A}, Γ) be a (not necessarily contractive) iterated function system. We call the correspondence (ϕ, E_Γ) the *Kajiwara-Watatani correspondence* associated to (\mathbb{A}, Γ) . We call the Cuntz-Pimsner algebra \mathcal{O}_{E_Γ} (see Appendix A.3) of (ϕ, E_Γ) the *Kajiwara-Watatani algebra* of (\mathbb{A}, Γ) . We denote \mathcal{O}_{E_Γ} by $C^*(\mathbb{A}, \Gamma)$.

In the case where (\mathbb{A}, Γ) is contractive and satisfies the open-set condition, Kajiwara and Watatani showed that $C^*(\mathbb{A}, \Gamma)$ is both simple and purely infinite [KW06, Theorem 3.7]. In this setting $C^*(\mathbb{A}, \Gamma)$ is separable, nuclear, and satisfies the Universal Coefficient Theorem. Accordingly, $C^*(\mathbb{A}, \Gamma)$ is classified by K -theory and the position of the unit by the Kirchberg-Phillips classification [Phi00]. We do not pursue classifiability of $C^*(\mathbb{A}, \Gamma)$ in the non-contractive setting.

To compute properties of the Cuntz-Pimsner algebra $C^*(\mathbb{A}, \Gamma)$ we need to understand the covariance ideal

$$I_{E_\Gamma} := \phi^{-1}(\text{End}_A^0(E_\Gamma))$$

of A . When (\mathbb{A}, Γ) is contractive and satisfies the open-set condition then it is shown in [KW06] that I_{E_Γ} is the ideal of continuous functions in A which vanish on the subset of \mathbb{A} consisting of those $x \in \mathbb{A}$ for which there exists $y \in \mathbb{A}$ and $\gamma \neq \gamma' \in \Gamma$ satisfying $x = \gamma(y) = \gamma'(y)$. This subset of \mathbb{A} is always a subset of the critical set C_Γ , but it is not usually the entire critical set.

We modify Kajiwara and Watatani's approach to handle our more general setting, and remove the hypothesis of the open-set condition.

Definition 2.2.10. For each non-empty subset $\Lambda \subseteq \Gamma$ consider the closed subset of $G_\Lambda \subseteq \text{Gr}(\Gamma)$ defined by $G_\Lambda := \bigcap_{\gamma \in \Lambda} \text{Gr}(\gamma)$. If $\Lambda \subseteq \Lambda'$ then $G_{\Lambda'} \subseteq G_\Lambda$. The *branched set*

of (\mathbb{A}, Γ) is defined by

$$B_\Gamma := \bigcup_{\Lambda \subseteq \Gamma} \partial G_\Lambda \subseteq \text{Gr}(\Gamma), \quad (2.8)$$

where ∂G_Λ denotes the boundary of G_Λ calculated in the topology of $\text{Gr}(\Gamma)$. Elements of B_Γ are called *branched points*.

Remark 2.2.11. In the union (2.8) we only need to consider Λ such that $|\Lambda| \geq 2$. To see why, suppose that $z \in \partial G_{\{\gamma\}} = \partial \text{Gr}(\gamma)$ for some $\gamma \in \Gamma$. Then for any open neighbourhood U of z there exists $\gamma' \neq \gamma$ and $w \in U \setminus \text{Gr}(\gamma)$. By passing to a subsequence if necessary, it follows that there exists $\gamma' \neq \gamma$ and a sequence $(w_n)_{n \in \mathbb{N}}$ in $\text{Gr}(\gamma')$ with $w_n \rightarrow z$. Since $\text{Gr}(\gamma')$ is closed, we have $z \in \text{Gr}(\gamma')$, so $z \in G_{\{\gamma, \gamma'\}}$. Moreover, $z \in \partial G_{\{\gamma, \gamma'\}}$ since an open neighbourhood of z intersects $\text{Gr}(\gamma) \setminus \text{Gr}(\gamma')$.

To see why the adjective “branched” is appropriate, consider the following example.

Example 2.2.12 (Clarence ^a). Let $\mathbb{A} = [0, 1]$ and let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$, where

$$\gamma_1(x) = \frac{x}{2}, \quad \gamma_2(x) = 1 - \frac{x}{2}, \quad \text{and} \quad \gamma_3(x) = \begin{cases} \frac{x}{4} + \frac{1}{8} & \text{if } x \in [0, \frac{1}{2}]; \\ \frac{x}{2} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Then (\mathbb{A}, Γ) is a contractive iterated function system with attractor $[0, 1]$ which does not satisfy the open-set condition.

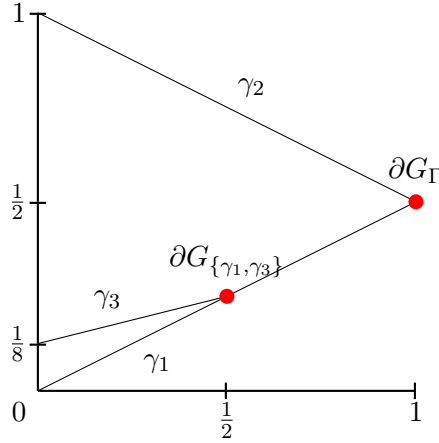


Figure 2.2: The graph of Clarence from Example 2.2.12. The branched points are highlighted in red.

The sets $\partial G_{\{\gamma_1, \gamma_3\}} = \{(1/4, 1/2)\}$ and $\partial G_\Gamma = \{(1/2, 1)\}$ are the only non-empty sets of the form ∂G_Λ for $|\Lambda| \geq 2$. Both are highlighted in Figure 2.2. ▲

Definition 2.2.13. For each $(x, y) \in \text{Gr}(\Gamma)$ define the *branch index* of (x, y) to be the number,

$$b(x, y) := \#\{\gamma \in \Gamma \mid x = \gamma(y)\}.$$

^aDuring the process of formulating the definition of the branched set, the author spent sufficient time with this example that we are now on a first-name basis.

The branch index was originally introduced in [KW06]. The branch index can also be described in terms of the sets $G_\Lambda = \bigcap_{\gamma \in \Lambda} \text{Gr}(\gamma)$ as

$$b(x, y) = \max\{|\Lambda| \mid \Lambda \subseteq \Gamma \text{ and } (x, y) \in G_\Lambda\}. \quad (2.9)$$

The branch index also gives an alternate description of the inner product on E_Γ . For each $\xi, \eta \in E_\Gamma$ we have,

$$(\xi \mid \eta)_A(y) = \frac{1}{|\Gamma|} \sum_{x \in \Gamma y} b(x, y) \overline{\xi(x, y)} \eta(x, y). \quad (2.10)$$

As such, for each $y \in \mathbb{A}$ the map $(x, y) \mapsto b(x, y)$ can be thought of as a weighted counting measure on the fibre $p_2^{-1}(y)$. This perspective is utilised when thinking of E_Γ as a topological quiver in Chapter 3 (see Example 3.1.12). The branched set can also be described in terms of the branch index.

Proposition 2.2.14. *Let (\mathbb{A}, Γ) be an iterated function system. Then*

$$B_\Gamma = \{(x, y) \in \text{Gr}(\Gamma) \mid b \text{ is discontinuous at } (x, y)\}.$$

Proof. Suppose that $(x, y) \in \text{Gr}(\Gamma) \setminus B_\Gamma$. Then there exists $\Lambda \subseteq \Gamma$ such that $(x, y) \in G_\Lambda$ but $(x, y) \notin G_{\Lambda'}$ for all $\Lambda \subsetneq \Lambda' \subseteq \Gamma$. Then $\text{int}(G_\Lambda) \setminus \bigcup_{\Lambda \subsetneq \Lambda' \subseteq \Gamma} G_{\Lambda'}$ is an open neighbourhood of (x, y) and according to Equation (2.9) we have $b(x', y') = b(x, y)$ for all $(x', y') \in \text{int}(G_\Lambda) \setminus \bigcup_{\Lambda \subsetneq \Lambda' \subseteq \Gamma} G_{\Lambda'}$. Hence, b is continuous at (x, y) .

Now suppose that b is continuous at $(x, y) \in \text{Gr}(\Gamma)$. Then there is an open neighbourhood U of (x, y) such that $b(x', y') = b(x, y)$ for all $(x', y') \in U$. Let $\Lambda \subseteq \Gamma$ be such that $(x, y) \in G_\Lambda$ and $b(x, y) = |\Lambda|$. Consider the open neighbourhood

$$V = U \setminus \bigcup \{G_{\Lambda'} \mid |\Lambda'| = |\Lambda| \text{ and } \Lambda' \neq \Lambda\}$$

of (x, y) . For any $(x', y') \in V$ we have $(x', y') \in G_\Lambda$ and since $b(x', y') = |\Lambda|$ we have $(x', y') \notin G_{\Lambda'}$ for any $\Lambda \subsetneq \Lambda' \subseteq \Gamma$. In particular, $V \subseteq \text{int}(G_\Lambda) \setminus \bigcup_{\Lambda \subsetneq \Lambda' \subseteq \Gamma} G_{\Lambda'}$. Now fix $(x', y') \in V$. Suppose for contradiction that $(x', y') \in \partial G_{\Lambda'}$ for some $\Lambda' \neq \Lambda$. Then $(x', y') \in G_{\Lambda'} \cap G_\Lambda \subseteq G_{\Lambda \cup \Lambda'}$. Since, $\Lambda \subsetneq \Lambda \cup \Lambda'$ this is impossible. Hence, $(x, y) \in V \subseteq \text{int}(G_\Lambda) \setminus \bigcup_{\Lambda \neq \Lambda'} \partial G_{\Lambda} \subseteq \text{Gr}(\Gamma) \setminus B_\Gamma$. \square

Corollary 2.2.15. *The map $b: \text{Gr}(\Gamma) \rightarrow \mathbb{N}$ is upper semi-continuous.*

Proof. Proposition 2.2.14 implies that b is continuous at each $(x, y) \in \text{Gr}(\Gamma) \setminus B_\Gamma$. So suppose that $(x, y) \in B_\Gamma$. Then $(x, y) \in \partial G_{\Lambda_0}$ for some $\Lambda_0 \subseteq \Gamma$ with $|\Lambda_0| \geq 2$. Without loss of generality we assume that Λ_0 is maximal in the sense that if $\Lambda_0 \subseteq \Lambda \subseteq \Gamma$ and $(x, y) \in G_\Lambda$ then $\Lambda = \Lambda_0$. Since G_Λ is closed for all $\Lambda \subseteq \Gamma$, it follows that

$$C = \bigcup \{G_\Lambda \mid \Lambda \subseteq \Gamma \text{ and } |\Lambda| \geq |\Lambda_0|\}$$

is a closed subset of $\text{Gr}(\Gamma)$. Maximality of Λ_0 implies that the open set $\text{Gr}(\Gamma) \setminus C$ contains (x, y) . Moreover, for each $(x', y') \in \text{Gr}(\Gamma) \setminus C$ we have $b(x', y') \leq |\Lambda| = b(x, y)$. Consequently, b is upper semi-continuous at (x, y) . \square

In [IKW07, p.1900] the authors introduced the notion of discontinuous points of b to determine the covariance ideal I_{E_Γ} . However, they erroneously claim that for an iterated

function system, the set of discontinuous points of b is equal to the set $\{(x, y) \in \text{Gr}(\Gamma) \mid x = \gamma(y) = \gamma'(y) \text{ for some } \gamma \neq \gamma'\}$. Their description is correct when (\mathbb{A}, Γ) is contractive and satisfies the open-set condition, which is likely to be the case they were considering.

Lemma 2.2.16. *If (\mathbb{A}, Γ) is contractive and satisfies the open-set condition, then*

$$\begin{aligned} B_\Gamma &= \{(x, y) \in \text{Gr}(\Gamma) \mid b(x, y) \geq 2\} \\ &= \{(x, y) \in \text{Gr}(\Gamma) \mid x = \gamma(y) = \gamma'(y) \text{ for some } \gamma \neq \gamma'\}. \end{aligned}$$

Proof. It is always true that if $(x, y) \in G_\Lambda$ for $|\Lambda| \geq 2$, then $b(x, y) \geq 2$. For the converse suppose that U is an open set realising the open-set condition. Suppose that $(x, y) \in \text{Gr}(\Gamma)$ satisfies $b(x, y) \geq 2$. Then there exist $\gamma_1, \gamma_2 \in \Gamma$ such that $x = \gamma_1(y) = \gamma_2(y)$. In particular, $(x, y) \in G_{\{\gamma_1, \gamma_2\}}$. The open-set condition now implies that $y \notin U$. Since U is dense and p_2 is both continuous and open, it follows that $p_2^{-1}(U)$ is a dense open subset of $\text{Gr}(\Gamma)$ such that $(x', y') \in p_2^{-1}(U)$ implies $b(x', y') = 1$. As such, for any open set W containing (x, y) in $\text{Gr}(\Gamma)$ we have $W \cap p_2^{-1}(U) \neq \emptyset$, and so there exists $(x', y') \in W$ with $b(x', y') = 1$. Consequently, $(x, y) \in \partial G_{\{\gamma_1, \gamma_2\}}$. \square

To characterise the covariance ideal $I_{E_\Gamma} = \phi^{-1}(\text{End}_A^0(E_\Gamma))$, we use the machinery developed by Muhly and Tomforde for topological quivers [MT05b]. Although topological quivers will not be encountered until Chapter 3, the correspondence E_Γ can be constructed as a C^* -correspondence of a topological quiver (see Example 3.1.12). The following Lemma is a reformulation of [MT05b, Corollary 3.12] for E_Γ .

Lemma 2.2.17. *Let (\mathbb{A}, Γ) be an iterated function system and suppose that $a \in A = C(\mathbb{A})$. Then $a \in A$ acts compactly on E_Γ —that is $a \in I_{E_\Gamma} = \phi^{-1}(\text{End}_A^0(E_\Gamma))$ —if and only if for every $x \in \mathbb{A}$ such that $a(x) \neq 0$ and every $y \in \mathbb{A}$ such that $x \in \Gamma y$, the projection $p_2: \text{Gr}(\Gamma) \rightarrow \mathbb{A}$ is locally injective at (x, y) .*

As Example 2.2.12 suggests, the local injectivity condition of Lemma 2.2.17 can be restated in terms of the branched set.

Lemma 2.2.18. *The map $p_2: \text{Gr}(\Gamma) \rightarrow \mathbb{A}$ is locally injective at $(x, y) \in \text{Gr}(\Gamma)$ if and only if $(x, y) \in \text{Gr}(\Gamma) \setminus B_\Gamma$.*

Proof. First suppose that $(x, y) \in B_\Gamma$. Let $\Lambda \subseteq \Gamma$ be such that $(x, y) \in \partial G_\Lambda$. Consider the open neighbourhood $(U \times V) \cap \text{Gr}(\Gamma)$ of (x, y) , where U, V are open subsets of \mathbb{A} . Since each $\gamma \in \Lambda$ is continuous, $W := V \cap \bigcap_{\gamma \in \Gamma} \gamma^{-1}(U)$ is an open neighbourhood of y . Moreover, $(U \times W) \cap \text{Gr}(\Gamma)$ is an open neighbourhood of (x, y) . Since $(x, y) \in \partial G_\Lambda$, there exists $(x', y') \in (U \times W) \cap \text{Gr}(\Gamma)$ such that $(x', y') \notin G_\Lambda$. In particular, there exist $\gamma_1, \gamma_2 \in \Lambda$ such that $\gamma_1(y') \neq \gamma_2(y')$. By construction of W , both $(\gamma_1(y'), y')$ and $(\gamma_2(y'), y')$ are elements of $(U \times W) \cap \text{Gr}(\Gamma)$. Since $p_2(\gamma_1(y'), y') = p_2(\gamma_2(y'), y')$ it follows that p_2 restricted to $(U \times V) \cap \text{Gr}(\Gamma)$ is not injective.

For the converse suppose that p_2 is not locally injective at (x, y) . Let $\Lambda = \{\gamma \in \Gamma \mid x = \gamma(y)\}$, so that $(x, y) \in G_\Lambda$. Let U be an open neighbourhood of (x, y) and consider the open neighbourhood $V := U \setminus (\bigcup_{\gamma \in \Gamma \setminus \Lambda} \text{Gr}(\gamma))$ of (x, y) , where we take the union to be empty if $\Gamma = \Lambda$. Since V is open there exist points $(x_1, y_1), (x_2, y_1) \in V$ such that $x_1 \neq x_2$. It follows from the construction of V that there exist $\gamma_1, \gamma_2 \in \Gamma$ such that $x_1 = \gamma_1(y_1)$ and $x_2 = \gamma_2(y_2)$. Since $x_1 \neq x_2$ we have $(x_1, y_1) \notin G_\Lambda$. Hence, $(x, y) \in \partial G_\Lambda \subseteq B_\Gamma$. \square

Combining Lemma 2.2.17 with Lemma 2.2.18 immediately yields the following characterisation of the covariance ideal I_{E_Γ} of E_Γ .

Proposition 2.2.19. *Let (\mathbb{A}, Γ) be an iterated function system. Then the covariance ideal of E_Γ is given by,*

$$I_{E_\Gamma} = C_0(\mathbb{A} \setminus p_1(B_\Gamma)).$$

Remark 2.2.20. If $B_\Gamma \neq \emptyset$, it follows that $\text{End}_A(E_\Gamma)$ is not isomorphic to $\text{End}_A^0(E_\Gamma)$. In particular, E_Γ is not finitely generated. For if E_Γ were finitely generated it would admit a finite frame $(e_i)_{i=1}^K$ satisfying $\sum_{i=1}^K \Theta_{e_i, e_i} = \text{id}_E = \phi(1_A)$.

We finish this section by showing that morphisms between iterated function systems induce $*$ -homomorphisms between Toeplitz algebras.

Proposition 2.2.21. *Let (\mathbb{A}, Γ) and (\mathbb{B}, Λ) be iterated function systems and suppose that $(f, \alpha): (\mathbb{A}, \Gamma) \rightarrow (\mathbb{B}, \Lambda)$ is a morphism with α bijective. Let (ϕ_Γ, E_Γ) and $(\phi_\Lambda, E_\Lambda)$ denote the corresponding Kajiwara-Watatani correspondences. Let $f^*: C(\mathbb{B}) \rightarrow C(\mathbb{A})$ denote the map dual to f and define $\psi: C(\text{Gr}(\Lambda)) \rightarrow C(\text{Gr}(\Gamma))$ by $\psi(\xi)(x_1, x_2) = \xi(f(x_1), f(x_2))$ for all $\xi \in C(\text{Gr}(\Lambda))$. Then (f^*, ψ) is a morphism of C^* -correspondences from $(\phi_\Lambda, E_\Lambda)$ to (ϕ_Γ, E_Γ) , and induces a $*$ -homomorphism $\Phi: \mathcal{T}_{E_\Lambda} \rightarrow \mathcal{T}_{E_\Gamma}$. Moreover, if (f, ψ) is a semiconjugacy, then (f^*, ψ) and Φ are injective.*

Proof. For each $\xi \in E_\Lambda$ and $a \in C(\mathbb{B})$ we have

$$\psi(\xi \cdot a)(x_1, x_2) = (\xi \cdot a)(f(x_1), f(x_2)) = \xi(f(x_1), f(x_2))a(f(x_2)) = (\psi(\xi) \cdot f^*(a))(x_1, x_2).$$

A similar calculation shows that the left action is also preserved. Since $\alpha(\gamma) \circ f = f \circ \gamma$ for all $\gamma \in \Gamma$, it follows that for all $\xi, \eta \in E_\Lambda$ we have

$$\begin{aligned} (\psi(\xi) \mid \psi(\eta))_{C(\mathbb{A})}(x_2) &= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{\xi(f \circ \gamma(x_2), f(x_2))} \eta(f \circ \gamma(x_2), f(x_2)) \\ &= \frac{1}{|\Lambda|} \sum_{\gamma \in \Gamma} \overline{\xi(\alpha(\gamma) \circ f(x_2), f(x_2))} \eta(\alpha(\gamma) \circ f(x_2), f(x_2)) \\ &= \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \overline{\xi(\lambda \circ f(x_2), f(x_2))} \eta(\lambda \circ f(x_2), f(x_2)) \\ &= (\xi \mid \eta)_{C(\mathbb{B})}(f(x_2)) \\ &= f^*(\xi \mid \eta)_{C(\mathbb{B})}(x_2). \end{aligned}$$

It follows that if (f, α) is a semiconjugacy, then $\|\psi(\xi)\| = \|\xi\|$ for all $\xi \in E_\Lambda$. The final statements follow from Lemma A.3.12. \square

Remark 2.2.22. Though morphisms of iterated function systems induce $*$ -homomorphisms between Toeplitz algebras, it is often not the case that the $*$ -homomorphism $\Phi: \mathcal{T}_{E_\Lambda} \rightarrow \mathcal{T}_{E_\Gamma}$ descends to a $*$ -homomorphism between the associated Cuntz-Pimsner algebras. Sometimes there are exceptions: see Proposition 2.5.17 and Proposition 2.5.29.

We mention that Dor-On [DO18] has introduced *weighted partial systems*, of which iterated function systems are an example. Dor-On has also introduced a notion of *graph conjugacy* and *branched transition conjugacy* between such systems and showed that two systems are branch-transition conjugate if and only if their associated tensor algebras

(non-self-adjoint operator algebras generated by creation operators) are isometrically isomorphic. Although we do not pursue the idea here, it would be interesting to see how these notions of conjugacy translate to the C^* -algebraic setting, since C^* -algebras have access to computable invariants like K -theory.

2.3 | Comparing $C^*(\mathbb{A}, \Gamma)$ and \mathcal{O}_{X_Γ}

We recall the following notion from [KW06].

Definition 2.3.1 ([KW06, p.18]). An iterated function system (\mathbb{A}, Γ) is said to be *graph separated* if $\text{Gr}(\gamma) \cap \text{Gr}(\gamma') = \emptyset$ for all $\gamma, \gamma' \in \Gamma$ with $\gamma \neq \gamma'$.

Graph separated systems are not uncommon. Example 1.1.10, Example 1.3.6, and Example 1.2.16 are all graph separated. If (\mathbb{A}, Γ) is graph separated, then B_Γ is necessarily empty.

If (\mathbb{A}, Γ) is graph separated then $\text{Gr}(\Gamma)$ is homeomorphic to $\mathbb{A} \times \Gamma$. Moreover, comparing the definition of E_Γ to X_Γ , we see that the map $\kappa: E_\Gamma \rightarrow X_\Gamma$ given by $\kappa(\xi)(x, \gamma) = |\Gamma|^{-\frac{1}{2}} \xi(\gamma(x), x)$ induces an isomorphism of correspondences.

Proposition 2.3.2. *Suppose that (\mathbb{A}, Γ) is graph separated. Then $C^*(\mathbb{A}, \Gamma) \cong \mathcal{O}_{X_E}$. If (\mathbb{A}, Γ) also admits a code map, then $C^*(\mathbb{A}, \Gamma)$ is isomorphic to the Cuntz algebra $\mathcal{O}_{|\Gamma|}$.*

Proof. The first statement follows from the isomorphism $\kappa: E_\Gamma \rightarrow X_\Gamma$, and the second follows from Proposition 2.1.4. \square

In light of Proposition 2.3.2 we see that for many iterated function systems, the Kajiwara-Watatani algebra $C^*(\mathbb{A}, \Gamma)$ contains the same information as \mathcal{O}_{X_Γ} . However, when (\mathbb{A}, Γ) has branched points, their existence presents many challenges which we explore throughout the remainder of this chapter as well as in Chapter 3.

One way to explain the difference between $C^*(\mathbb{A}, \Gamma)$ and \mathcal{O}_{X_Γ} is to think about the way they are constructed from a dynamical perspective. For this we introduce the notion of paths.

Definition 2.3.3. For each $k \in \mathbb{N}$ define,

$$\mathbb{A}^{(k)} := \{(x_1, \dots, x_k) \in \mathbb{A}^k \mid \forall 1 \leq i \leq k, x_i \in \Gamma x_{i+1}\},$$

which we endow with the subspace topology inherited from \mathbb{A}^k . We call $\mathbb{A}^{(k)}$ the space of *paths of length $k - 1$* in (\mathbb{A}, Γ) . For each $k \in \mathbb{N}$ we call $\Gamma^k \times \mathbb{A}$ the space of *labelled paths of length k* in (\mathbb{A}, Γ) .

Note that $\mathbb{A}^{(1)} = \mathbb{A}$ and $\mathbb{A}^{(2)} = \text{Gr}(\Gamma)$. The tensor powers of E_Γ and X_Γ can be realised in terms of paths.

Proposition 2.3.4 (cf. [KW06, Proposition 2.2]). *Let (\mathbb{A}, Γ) be an iterated function system. For each $k \in \mathbb{N}$ there is an isomorphism $\Psi_E: E^{\otimes k} \rightarrow C(\mathbb{A}^{(k+1)})$ of Banach spaces satisfying*

$$\Psi_E(\xi_1 \otimes \dots \otimes \xi_k)(x_1, \dots, x_{k+1}) = \xi_1(x_1, x_2) \cdots \xi_k(x_k, x_{k+1}),$$

for all $\xi_1, \dots, \xi_k \in E_\Gamma$. For each $k \in \mathbb{N}$ there is an isomorphism $\Psi_X: X_\Gamma^{\otimes k} \rightarrow C(\Gamma^k \times \mathbb{A})$ of Banach spaces satisfying

$$\begin{aligned} \Psi_X(\xi_1 \otimes \dots \otimes \xi_{k-1} \otimes \xi_k)(\gamma_1, \dots, \gamma_{k-1}, \gamma_k, x) \\ = \xi_1(\gamma_1, (\gamma_2 \circ \dots \circ \gamma_k)(x)) \dots \xi_{k-1}(\gamma_{k-1}, \gamma_k(x)) \xi_k(\gamma_k, x), \end{aligned}$$

for all $\xi_1, \dots, \xi_k \in X_\Gamma$.

Proof. The first statement is given by [KW06, Proposition 2.2]. The second follows from a similar Stone-Weierstrass style of argument. \square

Remark 2.3.5. Both $C(\mathbb{A}^{(k+1)})$ and $C(\Gamma^k \times \mathbb{A})$ can be equipped with the structure of an A - A -correspondence in such a way that the isomorphisms Ψ_E and Ψ_X become isomorphisms of A - A -correspondences (see [KW06, Proposition 2.2]). Since we do not use the A - A -correspondence structures, we choose to omit introducing them.

Recall from Appendix A that Toeplitz algebras can be represented concretely as C^* -algebras generated by creation and annihilation operators on a Fock module. For the two correspondences we have considered, the Fock modules are given by

$$\mathcal{F}(E_\Gamma) = \bigoplus_{k=0}^{\infty} E_\Gamma^{\otimes k} \quad \text{and} \quad \mathcal{F}(X_\Gamma) = \bigoplus_{k=0}^{\infty} X_\Gamma^{\otimes k}.$$

The Cuntz-Pimsner algebra is then a quotient of the Toeplitz algebra. In light of Proposition 2.3.4, we have the following heuristic:

- E_Γ and $C^*(\mathbb{A}, \Gamma)$ encode the dynamics of *paths* in (\mathbb{A}, Γ) ,
- X_Γ and \mathcal{O}_{X_Γ} encode the dynamics of *labelled paths* in (\mathbb{A}, Γ) .

For each $k \in \mathbb{N}$ there is a continuous surjection $F_k: \Gamma^k \times \mathbb{A} \rightarrow \mathbb{A}^{(k+1)}$ given by forgetting labels. That is,

$$F_k(\gamma_1, \dots, \gamma_{k-1}, \gamma_k, x) = ((\gamma_1 \circ \dots \circ \gamma_k)(x), \dots, (\gamma_{k-1} \circ \gamma_k)(x), \gamma_k(x), x).$$

For convenience, we denote F_1 by F . For $(x_1, x_2) \in \text{Gr}(\Gamma)$ we have $|F^{-1}(x_1, x_2)| > 1$ if and only if there is some $\Lambda \subseteq \Gamma$ with $|\Lambda| > 2$ such that $x \in G_\Lambda = \bigcap_{\gamma \in \Lambda} \text{Gr}(\gamma)$. In particular, (\mathbb{A}, Γ) is graph separated if and only F is bijective.

Recall that when (\mathbb{A}, Γ) is graph separated Proposition 2.1.4 yields an isomorphism $\kappa: E_\Gamma \rightarrow X_\Gamma$. Then $\kappa(\xi)(\gamma, x) = |\Gamma|^{-\frac{1}{2}} \xi(F(\gamma, x))$. It follows that forgetting labels induces an inclusion of correspondences.

Proposition 2.3.6. *Let (\mathbb{A}, Γ) be an iterated function system. The pair (id, κ) defines an injective morphism of correspondences from (ϕ, E_Γ) to (ϕ, X_Γ) . In particular, there is an injective $*$ -homomorphism $\Phi: \mathcal{T}_{E_\Gamma} \rightarrow \mathcal{T}_{X_\Gamma}$.*

Proof. That (id, κ) defines a morphism follows almost immediately from the definitions of X_Γ and E_Γ . The $*$ -homomorphism now follows from Lemma A.3.12. \square

Remark 2.3.7. Although there is an injective $*$ -homomorphism $\Phi: \mathcal{T}_{E_\Gamma} \rightarrow \mathcal{T}_{X_\Gamma}$ induced by forgetting labels, it is not clear whether Φ descends to a $*$ -homomorphism between the associated Cuntz-Pimsner algebras. In particular, it is not clear whether the morphism (id, κ) is covariant when (\mathbb{A}, Γ) is not graph separated.

2.4 | Examples

In this section we examine $C^*(\mathbb{A}, \Gamma)$ for some concrete classes of iterated function systems. Before we begin, observe that Proposition 2.2.19 together with Theorem A.3.17 yield the following.

Corollary 2.4.1. *Let (\mathbb{A}, Γ) be an iterated function system. Then there is a six-term exact sequence of Abelian groups,*

$$\begin{array}{ccccc} K_0(C_0(\mathbb{A} \setminus p_1(B_\Gamma))) & \xrightarrow{\otimes(\iota_{I, A^*} - [E_\Gamma])} & K_0(C(\mathbb{A})) & \xrightarrow{\iota_*} & K_0(C^*(\mathbb{A}, \Gamma)) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(C^*(\mathbb{A}, \Gamma)) & \xleftarrow{\iota_*} & K_1(C(\mathbb{A})) & \xleftarrow{\otimes(\iota_{I, A^*} - [E_\Gamma])} & K_1(C_0(\mathbb{A} \setminus p_1(B_\Gamma))) \end{array} .$$

The increased generality of Corollary 2.4.1 in comparison to [KW06] increases the number of examples of systems for which the K -theory of $C^*(\mathbb{A}, \Gamma)$ is computable.

2.4.1 Interval maps

Before we begin, we mention that C^* -algebraic approaches to interval dynamics have been considered previously by Shultz and Deaconu [Shu05; DS07]. Their approach uses a groupoid construction and in general differs from the Kajiwara-Watatani algebra as mentioned in the introduction of [DS07]. Similar considerations were made by Johannesen for circle dynamics [Joh17].

Suppose that (\mathbb{A}, Γ) is an iterated function system with $\mathbb{A} = [0, 1]$. It follows from Proposition 2.1.4 implies that if $B_\Gamma = \emptyset$, then $C^*(\mathbb{A}, \Gamma)$ is isomorphic to the Cuntz algebra $\mathcal{O}_{|\Gamma|}$. Accordingly $K_0(C^*(\mathbb{A}, \Gamma)) \cong \mathbb{Z}/(1 - |\Gamma|)\mathbb{Z}$ and $K_1(C^*(\mathbb{A}, \Gamma)) = 0$ (see [Cun81]). Hence, we restrict our attention to systems for which $B_\Gamma \neq \emptyset$, and compute the K -theory of $C^*(\mathbb{A}, \Gamma)$.

Proposition 2.4.2. *Let $([0, 1], \Gamma)$ be an iterated function system with $B_\Gamma \neq \emptyset$. Let $1 < M \leq \infty$ be the number of connected components of $[0, 1] \setminus p_1(B_\Gamma)$ and let*

$$\Delta = \begin{cases} 0 & \text{if } |\{0, 1\} \cap p_1(B_\Gamma)| = 2; \\ 1 & \text{if } |\{0, 1\} \cap p_1(B_\Gamma)| = 1; \\ 2 & \text{if } |\{0, 1\} \cap p_1(B_\Gamma)| = 0. \end{cases}$$

Then,

$$K_0(C^*([0, 1], \Gamma)) \cong \bigoplus_{i=1}^{M-\Delta+1} \mathbb{Z} \quad \text{and} \quad K_1(C^*([0, 1], \Gamma)) = 0.$$

The number $M - \Delta$ is an isomorphism invariant of $C^*([0, 1], \Gamma)$.

Proof. We make use of Corollary 2.4.1. Since $p_1(B_\Gamma)$ is closed $[0, 1] \setminus p_1(B_\Gamma)$ is open. Partition $[0, 1] \setminus p_1(B_\Gamma)$ into its connected components. Since the connected components are open in $[0, 1]$ and connected, they are open intervals (relative to $[0, 1]$). There are at most countably many connected components since they are disjoint and each contains a

rational number. Thus, there exists $1 < M \leq \infty$ mutually disjoint open intervals I_k such that $[0, 1] \setminus p_1(B_\Gamma) = \bigsqcup_{i=1}^M I_k$.

Observe that Δ is the number of intervals of the form $[0, a)$ or $(a, 1]$ in the decomposition of $[0, 1] \setminus p_1(B_\Gamma)$. Recall that $C_0([0, 1])$ has trivial K -groups. It now follows that $K_*(C_0([0, 1] \setminus p_1(B_\Gamma))) \cong \bigoplus_{i=1}^{M-\Delta} K_*(C_0(0, 1))$. In particular, $K_0(C_0([0, 1] \setminus p_1(B_\Gamma))) = 0$ and $K_1(C_0([0, 1] \setminus p_1(B_\Gamma))) \cong \bigoplus_{i=1}^{M-\Delta} \mathbb{Z}$.

Since $K_0(C([0, 1])) \cong \mathbb{Z}$ and $K_1(C([0, 1])) = 0$, the six-term sequence of Corollary 2.4.1 implies that

$$K_1(C^*([0, 1], \Gamma)) = 0 \quad \text{and} \quad K_0(C^*([0, 1], \Gamma)) \cong K_0(C([0, 1])) \oplus K_1(C_0([0, 1] \setminus p_1(B_\Gamma))).$$

The result then follows. \square

Example 2.4.3. Let $([0, 1], \Gamma)$ and $([0, 1], \Gamma')$ be the iterated function systems of Example 2.2.3. Then $C^*([0, 1], \Gamma)$ and $C^*([0, 1], \Gamma')$ are not isomorphic since $K_0(C^*([0, 1], \Gamma)) = 0$ while $K_0(C^*([0, 1], \Gamma')) \cong \mathbb{Z}$. Similarly, if $([0, 1], \Gamma'')$ is the system from Example 2.2.12, then $K_0(C^*([0, 1], \Gamma'')) \cong \mathbb{Z}^3$. \blacktriangle

In the statement of Proposition 2.4.2 it is tempting to try to relate the number $M - \Delta$ to the cardinality of the set $p_1(B_\Gamma)$. However, as the following example shows the set $p_1(B_\Gamma)$ can be uncountable, even when $([0, 1], \Gamma)$ is contractive.

Example 2.4.4. Let $\mathbb{A} = [0, 1]$ and $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. The maps γ_1 and γ_2 are given by $\gamma_1(x) = x/2$ and $\gamma_2(x) = 1 - x/2$. To define γ_3 , let $C \subseteq [0, 1]$ denote the middle-thirds Cantor set. Write the complement $[0, 1] \setminus C$ as a countable union of disjoint open intervals $[0, 1] \setminus C = \bigsqcup_{k \in \mathbb{N}_+} I_k$, where the intervals I_k are the middle-thirds which are removed in the construction of C . For each $k \in \mathbb{N}$ let $l_k, r_k \in [0, 1]$ be such that $I_k = (l_k, r_k)$. Fix $m \in (1/2, 1)$. Define a strict contraction γ_3 on $[0, 1]$ by,

$$\gamma_3(x) = \begin{cases} \frac{x}{2} & \text{if } x \in C; \\ mx + l_k(\frac{1}{2} - m) & \text{if } x \in (l_k, \frac{l_k + r_k}{2}); \\ (1 - m)x - r_k(\frac{1}{2} - m) & \text{if } x \in (\frac{l_k + r_k}{2}, r_k). \end{cases}$$

The iterated function system Γ is perhaps best understood via its graph (Figure 2.3).

We claim that B_Γ is homeomorphic to C . Clearly p_2 is not locally injective at $(1/2, 1)$. Fix $x \in C \setminus \{1\}$ and consider the point $(x/2, x) \in \text{Gr}(\Gamma)$. Suppose that U is an open neighbourhood of $(x/2, x)$ in $\text{Gr}(\Gamma)$. Since $\text{Gr}(\gamma_1)$ is closed we can assume, without loss of generality, that $U \cap \text{Gr}(\gamma_1) = \emptyset$.

Since $[0, 1] \setminus C$ is dense in $[0, 1]$ there exists a sequence $(x_m)_{m=1}^\infty$ in $[0, 1] \setminus C$ such that $x_m \rightarrow x$. Since γ_2 and γ_3 are both continuous, $((\gamma_2(x_m), x_m))_{m=1}^\infty$ and $((\gamma_3(x_m), x_m))_{m=1}^\infty$ converge in $\text{Gr}(\Gamma)$ to $(x/2, x)$. Hence, there exists $M \in \mathbb{N}$ such that $(\gamma_2(x_M), x_M)$ and $(\gamma_3(x_M), x_M)$ both belong to U . In particular, p_2 is not locally injective at $(x/2, x)$ so $(x/2, x) \in B_\Gamma$.

Conversely, if $x \in [0, 1] \setminus C$, then $\text{Gr}(\Gamma) \setminus (\text{Gr}(\gamma_3) \cup \text{Gr}(\gamma_1))$ is an open neighbourhood of $(\gamma_2(x), x)$, $\text{Gr}(\Gamma) \setminus (\text{Gr}(\gamma_2) \cup \text{Gr}(\gamma_1))$ is an open neighbourhood of $(\gamma_3(x), x)$, and the restriction of p_2 to each of these neighbourhoods is injective. Hence, $B_\Gamma = \{(x/2, x) \mid x \in C\}$. Since p_1 restricts to a homeomorphism from $\text{Gr}(\gamma_2)$ to $[0, 1/2]$, and $B_\Gamma \subseteq \text{Gr}(\gamma_2)$, it follows that $p_1(B_\Gamma)$ is also homeomorphic to C . In particular, $p_1(B_\Gamma)$ is uncountable. \blacktriangle

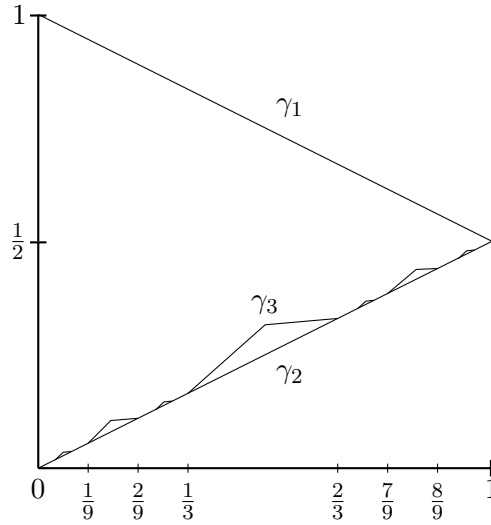


Figure 2.3: The graph of $([0, 1], \Gamma)$ from Example 2.4.4 with a value of m close to 1. The graph is only drawn to the third iterate of the middle-thirds Cantor set.

2.4.2 Sierpinski gasket

Let \mathbb{A} denote the Sierpinski gasket with vertices $(0, 0)$, $(1/2, \sqrt{3}/2)$, and $(1, 0)$.

Example 2.4.5. Let $(\mathbb{A}, \Gamma = \{\gamma_1, \gamma_2, \gamma_3\})$ be the iterated function system from Example 1.1.10, with

$$\gamma_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right), \quad \gamma_2(x, y) = \left(\frac{x+1}{4}, \frac{2y+\sqrt{3}}{4}\right), \quad \text{and} \quad \gamma_3(x, y) = \left(\frac{x+1}{2}, \frac{y}{2}\right).$$

The system (\mathbb{A}, Γ) is graph separated, so Proposition 2.3.2 implies that $C^*(\mathbb{A}, \Gamma)$ is isomorphic to the Cuntz algebra \mathcal{O}_3 . Hence, $K_0(C^*(\mathbb{A}, \Gamma)) \cong \mathbb{Z}/2\mathbb{Z}$ while $K_1(C^*(\mathbb{A}, \Gamma)) = 0$. \mathbb{A}

Example 2.4.6. Following [KW05, Example 4.6] we can modify the system (\mathbb{A}, Γ) of Example 2.4.5 to arrive at a system which is not branch separated. To this end, let $R_{\frac{2\pi}{3}}$ denote the anti-clockwise rotation by $2\pi/3$ about the origin. Define $\gamma'_1: \mathbb{A} \rightarrow \mathbb{A}$ by

$$\gamma'_1(x, y) = R_{\frac{2\pi}{3}} \circ \gamma_1(x, y) + \left(\frac{1}{2}, 0\right).$$

Then $(\mathbb{A}, \Gamma' = \{\gamma'_1, \gamma_2, \gamma_3\})$ is a contractive iterated function system with attractor \mathbb{A} and satisfies the open-set condition. Moreover, $\gamma'_1(0, 0) = (1/2, 0) = \gamma_3(0, 0)$, and it follows that $p_1(B_{\Gamma'}) = \{(1/2, 0)\}$. Hence, the covariance ideal $I_{\Gamma'}$ is equal to $C_0(\mathbb{A} \setminus \{(1/2, 0)\})$ by Proposition 2.2.19.

In Corollary 4.3.10, we will see that $K_0(C(\mathbb{A})) = \mathbb{Z}[1_{C(\mathbb{A})}]$. Since the one-point compactification of $\mathbb{A} \setminus \{(1/2, 0)\}$ is nothing but \mathbb{A} , it follows that $K_0(I_{\Gamma'}) = 0$. The six-term sequence of Corollary 2.4.1 therefore implies that $K_0(C^*(\mathbb{A}, \Gamma'))$ contains a subgroup isomorphic to \mathbb{Z} . Consequently, $C^*(\mathbb{A}, \Gamma')$ is not isomorphic to $C^*(\mathbb{A}, \Gamma)$. \mathbb{A}

2.4.3 Square dynamics

There are many examples of iterated function systems on the unit square $\square := [0, 1] \times [0, 1]$, and we could not hope for a complete classification of K -groups like we did for interval maps in Section (2.4.1). Instead we present a few examples in order to highlight possible structures of the branched set.

Example 2.4.7. Consider the iterated function system $(\square, \Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\})$, where

$$\begin{aligned} \gamma_1(x, y) &= \left(\frac{x}{2}, \frac{y}{2}\right), & \gamma_2(x, y) &= \left(1 - \frac{x}{2}, \frac{y}{2}\right), \\ \gamma_3(x, y) &= \left(\frac{x}{2}, 1 - \frac{y}{2}\right), \text{ and } & \gamma_4(x, y) &= \left(1 - \frac{x}{2}, 1 - \frac{y}{2}\right). \end{aligned}$$

To get a feel for the structure of $\text{Gr}(\Gamma)$ observe that for $y \in [0, 1) \times [0, 1)$ we have $|p_2^{-1}(y)| = 4$, while for $y \in (\{1\} \times [0, 1)) \cup ([0, 1) \times \{1\})$ we have $|p_2^{-1}(y)| = 2$, and $|p_2^{-1}(1, 1)| = 1$.

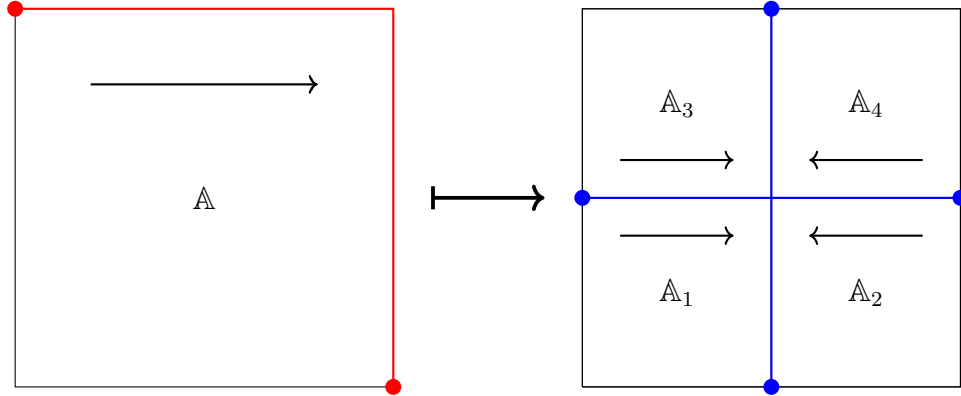


Figure 2.4: The iterated function system of Example 2.4.7. The internal arrows indicate the orientation of the squares. The set $p_2(B_\Gamma)$ is highlighted in red, while $p_1(B_\Gamma)$ is blue.

It is straightforward to verify that (\square, Γ) satisfies the open-set condition with open set $V = (0, 1) \times (0, 1)$ witnessing the condition. Lemma 2.2.16 therefore implies that $B_\Gamma = \{(x, y) \in \text{Gr}(\Gamma) \mid b(x, y) \geq 2\}$. In particular, $p_1(B_\Gamma)$ can be identified with the cross in the centre of the square on the right-hand side of Figure 2.4. That is $p_1(B_\Gamma) = (\{1/2\} \times [0, 1]) \cup ([0, 1] \times \{1/2\})$.

The subspace $\square \setminus p_1(B_\Gamma)$ is homeomorphic to the disjoint union of 4 copies of $[0, 1) \times [0, 1)$. Since the one-point compactification of $[0, 1) \times [0, 1)$ is homeomorphic to the disk \mathbb{D} , it follows that $K_0(I_\Gamma) = K_1(I_\Gamma) = 0$. The six-term sequence of Corollary (2.4.1) now implies that

$$K_0(C^*(\square, \Gamma)) = \mathbb{Z}[1_{C^*(\square, \Gamma)}] \quad \text{and} \quad K_1(C^*(\square, \Gamma)) = 0.$$

In particular, $C^*(\square, \Gamma)$ is not isomorphic to \mathcal{O}_4 . △

Example 2.4.8. Let $X = [1/4, 3/4] \times [1/4, 3/4] \subset \square$. Define an iterated function system $(\square, \Gamma = \{\gamma_1, \gamma_2\})$ by letting $\gamma_1 = \text{id}_\square$ and letting $\gamma_2: \square \rightarrow \square$ be any continuous injective function such that $\gamma_2(x, y) = (x, y)$ for all $(x, y) \in X$, and $\gamma_2(x, y) \neq (x, y)$ for all $(x, y) \in \square \setminus X$.

By construction $G_{\{\gamma_1, \gamma_2\}} = \{(q, q) \mid q \in X\}$, and it follows that $p_1(B_\Gamma) = \partial X$. In particular, $p_1(B_\Gamma)$ is homeomorphic to a circle. The space $\square \setminus p_1(B_\Gamma)$ is the disjoint union of $\square \setminus X$ and $\text{int}(X)$. Since the one point compactification of $\square \setminus X$ is a disk, it follows that $K_0(C_0(\square \setminus X)) = K_1(C_0(\square \setminus X)) = 0$. On the other hand, $\text{int}(X)$ is homeomorphic to \mathbb{R}^2 . Let $p_{\text{bott}} \in M_2(C(S^2))$ denote the Bott projection (see [GBVF01, §2.6]). Then $K_0(C_0(\text{int}(X))) \cong \mathbb{Z}$ which is generated by the class $[p_{\text{bott}}] - [1]$ in $K_0(C(S^2))$. We also have $K_1(C_0(\text{int}(X))) = 0$. The six-term sequence of Corollary (2.4.1) reads as,

$$\begin{array}{ccccc} K_0(C_0(\text{int}(X))) \cong \mathbb{Z} & \xrightarrow{\otimes(\iota_{I, A^*} - [E_\Gamma])} & K_0(C(\square)) \cong \mathbb{Z} & \xrightarrow{\iota_*} & K_0(C^*(\square, \Gamma)) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(C^*(\square, \Gamma)) & \longleftarrow & 0 & \longleftarrow & 0 \end{array} \quad .$$

By construction, $\cdot \otimes [E_\Gamma]: K_0(C_0(\text{int}(X))) \rightarrow K_0(C(\square))$ is nothing but the map induced by the inclusion $\iota_{C_0(\text{int}(X)), A^*}$. Consequently, $\cdot \otimes (\iota_{I, A^*} - [E_\Gamma])$ is the zero map. It therefore follows that

$$K_0(C^*(\square, \Gamma)) = \mathbb{Z}[1_{C^*(\square, \Gamma)}] \quad \text{and} \quad K_1(C^*(\square, \Gamma)) \cong \mathbb{Z}([p_{\text{bott}}] - [1]).$$

This is the first example we have considered for which the Kajiwara-Watatani algebra has a non-trivial K_1 -group. \triangleleft

2.5 | Invertible systems

In this section we look at the class of invertible iterated function systems and their Kajiwara-Watatani algebras.

Definition 2.5.1 ([Kie02, Definition 5.4.6]). Let (\mathbb{A}, Γ) be an iterated function system. We say that (\mathbb{A}, Γ) is *invertible* if it is injective and $\gamma^{-1}(x) = \gamma'^{-1}(x)$ for all $x \in \gamma(\mathbb{A}) \cap \gamma'(\mathbb{A})$ and $\gamma, \gamma' \in \Gamma$.

Invertibility can be rephrased by saying that every critical point of (\mathbb{A}, Γ) is the image of a unique point under Γ . Indeed, if $x \in C_\Gamma$ (see Definition 1.2.12), then there exist $\gamma \neq \gamma'$ such that $x \in \gamma(\mathbb{A}) \cap \gamma'(\mathbb{A})$. Invertibility now implies that $x = \gamma(y) = \gamma'(y)$ for some $y \in \mathbb{A}$. We can use the existence of such a y to define a continuous map on \mathbb{A} which acts as a left inverse to the maps $\gamma \in \Gamma$.

Proposition 2.5.2 ([Kie02, Proposition 5.4.3 (i)]). *Suppose that (\mathbb{A}, Γ) is an invertible iterated function system. Then there is a well-defined continuous surjection $\sigma: \mathbb{A} \rightarrow \mathbb{A}$ given by*

$$\sigma(x) = \gamma^{-1}(x), \quad x \in \gamma(\mathbb{A}).$$

Moreover, for each $\gamma \in \Gamma$ we have $\sigma \circ \gamma = \text{id}_{\mathbb{A}}$.

Proof. Well-definedness follows from the definition of invertibility. Surjectivity follows from the fact that $\mathbb{A} = \bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A})$. For continuity note that $\sigma^{-1}(C) = \bigcup_{\gamma \in \Gamma} \gamma(C)$ is closed for every closed set $C \subseteq \mathbb{A}$. The final statement follows immediately from the definition of γ . \square

Definition 2.5.3. Let (\mathbb{A}, Γ) be an invertible iterated function system. We call the continuous surjection $\sigma: \mathbb{A} \rightarrow \mathbb{A}$ of Proposition 2.5.2, which satisfies $\sigma \circ \gamma = \text{id}$ for all $\gamma \in \Gamma$, the *shift map* associated to (\mathbb{A}, Γ) .

The name shift map comes from [Kie02, Definition 5.4.6]. Although σ does not immediately look like a shift in the traditional sense, we will see in Proposition 2.5.22 that every invertible iterated function system is conjugate to a space of sequences for which σ coincides with the left-shift.

Invertible iterated function systems were implicitly considered in [KW06], with Γ being referred to as the inverse branches σ . We have already seen some examples of invertible systems.

Example 2.5.4. Consider the code space $(\Omega_N, \bar{\Gamma})$. Then $(\Omega_N, \bar{\Gamma})$ is invertible with shift map given by the left-shift $\sigma: \Omega_N \rightarrow \Omega_N$ satisfying $\sigma(w_1 w_2 w_3 \cdots) = w_2 w_3 \cdots$. \triangleleft

Example 2.5.5. Let (\mathbb{A}, Γ') be as in Example 2.2.3. Then (\mathbb{A}, Γ') is invertible with shift map given by the tent map. On the other hand, (\mathbb{A}, Γ) from Example 2.2.3 is not invertible since $\gamma_1^{-1}(1/2) \neq \gamma_2^{-1}(1/2)$. \triangleleft

Example 2.5.6. The system $(\square, \Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\})$ of Example 2.4.7 is invertible, with shift map $\sigma: \mathbb{A} \rightarrow \mathbb{A}$ given by

$$\sigma(x, y) = \begin{cases} (2x, 2y) & \text{if } 0 \leq x, y \leq \frac{1}{2}; \\ (2(1-x), 2y) & \text{if } 0 \leq y \leq \frac{1}{2} \leq x \leq 1 \\ (2x, 2(1-y)) & \text{if } 0 \leq x \leq \frac{1}{2} \leq y \leq 1 \\ (2(1-x), 2(1-y)) & \text{if } \frac{1}{2} \leq x, y \leq 1. \end{cases}$$

The map σ could be considered a 2-dimensional version of the tent map from Example 2.2.3. \triangleleft

If (\mathbb{A}, Γ) is an invertible iterated function system, then the graph of Γ takes the form,

$$\text{Gr}(\Gamma) = \{(x, \sigma(x)) \mid x \in \mathbb{A}\}.$$

In particular, the graph of Γ can be realised as the (transpose of) the graph of σ . We obtain the following result.

Lemma 2.5.7. *Let (\mathbb{A}, Γ) be an invertible iterated function system. Then the projection $p_1: \text{Gr}(\Gamma) \rightarrow \mathbb{A}$ given by $p_1(x, \sigma(x)) = x$ is a homeomorphism*

Proof. The map p_1 defines continuous bijection from the compact set $\text{Gr}(\Gamma)$ to the Hausdorff space \mathbb{A} . Therefore, p_1 is a homeomorphism. \square

Using Lemma 2.5.7 we give an alternate description for the Kajiwara-Watatani correspondence for invertible systems.

Proposition 2.5.8. *Suppose that (\mathbb{A}, Γ) is an invertible iterated function system. Let (id, A_σ) be the A - A -correspondence with $A_\sigma = A$ and the following structures:*

- the right action of $a \in A$ on $\xi \in A_\sigma$ is given by, $(\xi \cdot a)(x) = \xi(x)a(\sigma(x))$;

- the A -valued inner product of $\xi, \eta \in A_\sigma$ is given by

$$(\xi \mid \eta)_A(x) = \frac{1}{|\Gamma|} \sum_{y \in \sigma^{-1}(x)} b(y, x) \overline{\xi(y)} \eta(y); \quad \text{and}$$

- the left action of A on A_σ is given by multiplication in A .

Then (id, p_1^*) defines an isomorphism of correspondences from (id, A_σ) to (ϕ, E_Γ) .

Proof. This is straightforward to check using Lemma 2.5.7. \square

Remark 2.5.9. If (\mathbb{A}, Γ) is invertible then we can identify B_Γ with $p_1(B_\Gamma)$ via the homeomorphism p_1 .

Remark 2.5.10. If (\mathbb{A}, Γ) is invertible then using the identification of (ϕ, E_Γ) with (id, A_σ) , we can equip E_Γ with a left A -valued inner product ${}_A(\xi \mid \eta) = \xi \eta^*$ for $\xi, \eta \in A_\sigma$. Since this inner product induces the supremum norm on A_σ , it follows from Lemma 2.2.8 that with this left inner product (ϕ, E_Γ) becomes a bi-Hilbertian A - A -bimodule (see [KPW04, Definition 2.3.]). In general though, (ϕ, E_Γ) is not an imprimitivity bimodule since $I_\Gamma \neq A$ when $B_\Gamma \neq \emptyset$.

In Section 2.5.2 we will show that every non-invertible iterated function system (\mathbb{A}, Γ) can be lifted to an invertible system $(\tilde{\mathbb{A}}, \tilde{\Gamma})$. As a stepping stone, we first introduce the concept of graph systems.

2.5.1 Graph systems

Let (\mathbb{A}, Γ) be an iterated function system. For each $\gamma \in \Gamma$ define $\gamma^{(2)}: \text{Gr}(\Gamma) \rightarrow \text{Gr}(\Gamma)$ by $\gamma^{(2)}(x_1, x_2) = (\gamma(x_1), x_1)$. Then $\gamma^{(2)}$ is continuous since γ is continuous. Since each $x_1 \in \mathbb{A}$ is in the image of some $\gamma \in \Gamma$ we have

$$\bigcup_{\gamma \in \Gamma} \gamma^{(2)}(\text{Gr}(\Gamma)) = \bigcup_{\gamma \in \Gamma} \text{Gr}(\gamma) = \text{Gr}(\Gamma).$$

As such, $(\text{Gr}(\Gamma), \Gamma^{(2)} = \{\gamma^{(2)} \mid \gamma \in \Gamma\})$ is an iterated function system. Even if (\mathbb{A}, Γ) is injective, $(\text{Gr}(\Gamma), \Gamma^{(2)})$ is typically not injective: if $(x_1, x_2), (x_1, x'_2) \in \text{Gr}(\Gamma)$, then for any $\gamma \in \Gamma$, $\gamma^{(2)}(x_1, x_2) = \gamma^{(2)}(x_1, x'_2)$.

Definition 2.5.11. We call $(\text{Gr}(\Gamma), \Gamma^{(2)} = \{\gamma^{(2)} \mid \gamma \in \Gamma\})$ the *graph system* associated to (\mathbb{A}, Γ) .

We remark that the graph systems we consider are a distinct from the graph directed iterated function systems of Mauldin and Williams [MW88]. As far as the author is aware, graph systems satisfying Definition 2.5.11 have not previously appeared in the literature.

In general, a graph system is not conjugate to the original system, as $\text{Gr}(\Gamma)$ is often not homeomorphic to \mathbb{A} . For example if (\mathbb{A}, Γ) is the system from Example 2.2.3 then $\text{Gr}(\Gamma)$ is homeomorphic to $\mathbb{A} \sqcup \mathbb{A}$. When (\mathbb{A}, Γ) is invertible however, the graph system coincides with the original system.

Lemma 2.5.12. Let (\mathbb{A}, Γ) be an iterated function system, and let $p_1: \text{Gr}(\Gamma) \rightarrow \mathbb{A}$ denote the projection onto the first factor. Then $(p_1, \gamma^{(2)} \mapsto \gamma)$ is a semiconjugacy from $(\text{Gr}(\Gamma), \Gamma^{(2)})$ to (\mathbb{A}, Γ) . If (\mathbb{A}, Γ) is invertible then (\mathbb{A}, Γ) is conjugate to $(\text{Gr}(\Gamma), \Gamma^{(2)})$ via $(p_1, \gamma^{(1)} \mapsto \gamma)$.

Proof. Clearly $\gamma \circ p_1 = p_1 \circ \gamma^{(2)}$ for all $\gamma \in \Gamma$. Since p_1 is surjective, it follows that $(p_1, \gamma^{(1)} \mapsto \gamma)$ is a semiconjugacy. The conjugacy statement follows from Lemma 2.5.7. \square

Since $(\text{Gr}(\Gamma), \Gamma^{(2)})$ is itself an iterated function system, we can consider its graph $\text{Gr}(\Gamma^{(2)})$. Let $p_1^{(2)}: \text{Gr}(\Gamma^{(2)}) \rightarrow \text{Gr}(\Gamma)$ and $p_2^{(2)}: \text{Gr}(\Gamma^{(2)}) \rightarrow \text{Gr}(\Gamma)$ denote the projections onto the first and second factors, respectively. Our aim is now to relate the branched structure of (\mathbb{A}, Γ) to the branched structure of the graph system $(\text{Gr}(\Gamma), \Gamma^{(2)})$. We first require the following result.

Lemma 2.5.13. *Let (\mathbb{A}, Γ) be an iterated function system. Let $x \in \mathbb{A}$ and suppose that $(x_n)_{n=1}^\infty$ is a sequence in \mathbb{A} with $x_n \rightarrow x$. Then there exists $\gamma \in \Gamma$ and a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $x \in \gamma(\mathbb{A})$ and $x_{n_k} \in \gamma(\mathbb{A})$ for all $k \in \mathbb{N}$.*

Proof. Let $\Lambda = \{\gamma \in \Gamma \mid x \in \gamma(\mathbb{A})\}$, which is non-empty since $\mathbb{A} = \bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A})$. Let U be an open neighbourhood of x . Then $V := U \setminus \bigcup_{\gamma \in \Gamma \setminus \Lambda} \gamma(\mathbb{A})$ is an open neighbourhood of x . Since $x_n \rightarrow x$, by passing to a subsequence we can assume that $x_n \in V$ for all $n \in \mathbb{N}$. Since $\mathbb{A} = \bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A})$ and $x_n \notin \gamma(\mathbb{A})$ for all $\gamma \in \Gamma \setminus \Lambda$, there exists some $\gamma \in \Lambda$ such that infinitely many x_n belong to $\gamma(\mathbb{A})$. Passing to the subsequence consisting of these elements gives the result. \square

We have the following relation between the branched structure of (\mathbb{A}, Γ) and its graph system.

Lemma 2.5.14. *Let (\mathbb{A}, Γ) be an iterated function system and consider its graph system $(\text{Gr}(\Gamma), \Gamma^{(2)})$. If $((x_1, x_2), (x_2, x_3)) \in B_{\Gamma^{(2)}}$, then $(x_1, x_2) \in B_\Gamma$. Conversely, if $(x_1, x_2) \in B_\Gamma$, then there exists $x_3 \in \mathbb{A}$ such that $((x_1, x_2), (x_2, x_3)) \in B_{\Gamma^{(2)}}$. In particular,*

$$I_{E_{\Gamma^{(2)}}} = C_0(\text{Gr}(\Gamma) \setminus B_\Gamma).$$

Proof. Fix $x = ((x_1, x_2), (x_2, x_3))$ in $\text{Gr}(\Gamma^{(2)})$. Suppose that p_2 is locally injective at (x_1, x_2) , and take an open neighbourhood U of (x_1, x_2) such that $p_2|_U$ is injective. Consider the open neighbourhood $V := (p_1^{(2)})^{-1}(U)$ of x in $\text{Gr}(\Gamma^{(2)})$, and suppose that $y, z \in V$ are such that $(y_2, y_3) = (z_2, z_3)$. Since (y_1, y_2) and (z_1, z_2) both belong to U and $y_2 = z_2$ it follows that $y_1 = z_1$. Hence, $p_2^{(2)}$ is injective when restricted to V so that $p_2^{(2)}$ is locally injective at x . Therefore, if $((x_1, x_2), (x_2, x_3)) \in B_{\Gamma^{(2)}}$, then $(x_1, x_2) \in B_\Gamma$.

Now suppose that $(x_1, x_2) \in B_\Gamma$. Since $p_2|_U$ is not injective for any open neighbourhood U of (x_1, x_2) , there exist sequences $(y_{n,1}, y_{n,2}) \rightarrow (x_1, x_2)$ and $(y'_{n,1}, y_{n,2}) \rightarrow (x_1, x_2)$ in $\text{Gr}(\Gamma)$ with $y_{n,1} \neq y'_{n,1}$. Using Lemma 2.5.13 we pass to a subsequence of $(y_{n,2})_{n=1}^\infty$ such that there exists $\gamma \in \Lambda$ with $x_2 \in \gamma(\mathbb{A})$ and $y_{n,2} \in \gamma(\mathbb{A})$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $z_n \in \gamma^{-1}(y_{n,2})$. Since \mathbb{A} is compact, we can pass to a convergent subsequence of $(z_n)_{n=1}^\infty$ whose limit we denote by z . As γ is continuous and $y_{n,2} \rightarrow x_2$ we have $\gamma(z) = x_2$.

Now consider $x := ((x_1, x_2), (x_2, z)) \in \text{Gr}(\Gamma^{(2)})$. Let U be an open neighbourhood of (x_1, x_2) and let V be an open neighbourhood of (x_2, z) in $\text{Gr}(\Gamma)$. Then $W := (U \times V) \cap \text{Gr}(\Gamma^{(2)})$ is a basic open neighbourhood of x in $\text{Gr}(\Gamma^{(2)})$. Fix $N \in \mathbb{N}$ such that $(y_{N,1}, y_{N,2}) \in U$, $(y'_{N,1}, y_{N,2}) \in U$ and $(y_{N,2}, z_N) \in V$. Then both $((y_{N,1}, y_{N,2}), (y_{N,2}, z_N))$ and $((y'_{N,1}, y_{N,2}), (y_{N,2}, z_N))$ belong to W . Consequently, $p_2^{(2)}$ is not locally injective at x . Hence, $((x_1, x_2), (x_2, z)) \in B_{\Gamma^{(2)}}$.

For the final statement note that $p_1^{(2)}(B_{\Gamma^{(2)}}) = B_\Gamma$ and apply Proposition 2.2.19. \square

The following example shows that if $(x_1, x_2) \in B_\Gamma$ it is not necessarily true that $((x_1, x_2), (x_2, x_3)) \in B_{\Gamma^{(2)}}$ for all $(x_2, x_3) \in \text{Gr}(\Gamma)$.

Example 2.5.15. Consider the contractive iterated function system $([0, 1], \Gamma = \{\gamma_1, \gamma_2, \gamma_3\})$, where

$$\gamma_1(x) = \frac{x}{2}, \quad \gamma_2(x) = \frac{x+1}{2}, \quad \text{and} \quad \gamma_3(x) = \begin{cases} \frac{x}{4} + \frac{1}{8} & \text{if } x \in [0, \frac{1}{2}]; \\ \frac{x}{2} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

The graph of Γ can be seen in Figure 2.5, and $B_\Gamma = \{(1/4, 1/2)\}$. Consider the graph

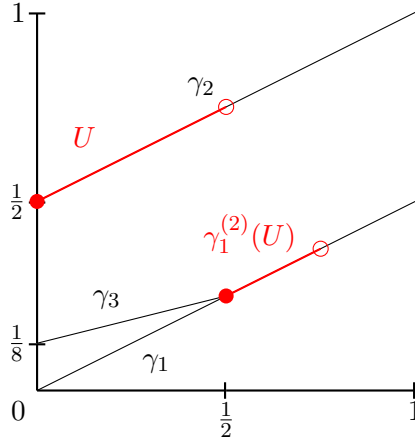


Figure 2.5: The graph of the iterated function system from Example 2.5.15. A choice of U and $\gamma_1^{(2)}(U)$ are highlighted.

system $(\text{Gr}(\Gamma), \Gamma^{(2)})$. In the graph system we have $\gamma_1^{(2)}(1/2, 0) = (1/4, 1/2)$. Let U be an open neighbourhood of $(1/2, 0)$ in $\text{Gr}(\Gamma)$ contained in $\text{Gr}(\gamma_2)$, as pictured in Figure 2.5 (we could take $U = \text{Gr}(\gamma_2)$). Then $\gamma_1^{(2)}(U)$ is not open. On the other hand, $V := (\text{Gr}(\{\gamma_1, \gamma_3\}) \times U) \cap \text{Gr}(\Gamma^{(2)})$ is an open neighbourhood of $((1/4, 1/2), (1/2, 0))$ in $\text{Gr}(\Gamma^{(2)})$. Moreover,

$$V = \{((\gamma_1(x_2), x_2), (x_2, x_3)) \mid (x_2, x_3) \in U\}.$$

Since $U \subseteq \text{Gr}(\gamma_2)$ it follows that $p_2^{(2)}$ is injective when restricted to V . Hence, $p_2^{(2)}$ is locally injective at $((1/4, 1/2), (1/2, 0))$, while $(1/4, 1/2) \in B_\Gamma$. \triangleleft

Remark 2.5.16. Let (\mathbb{A}, Γ) be an iterated function system with graph system $(\text{Gr}(\Gamma), \Gamma^{(2)})$. Recall that $G_\Lambda = \bigcap_{\gamma \in \Lambda} \text{Gr}(\gamma)$ for $\Lambda \subseteq \Gamma$. The critical set of the graph system is given by,

$$C_{\Gamma^{(2)}} = \bigcup_{\gamma \neq \gamma' \in \Gamma} \gamma^{(2)}(\text{Gr}(\Gamma)) \cap \gamma'^{(2)}(\text{Gr}(\Gamma)) = \bigcup \{G_\Lambda \mid \Lambda \subseteq \Gamma, |\Lambda| \geq 2\}.$$

It follows from the definition of the branched set that B_Γ is always contained in the critical set of the graph system $(\text{Gr}(\Gamma), \Gamma^{(2)})$.

Elements of $\text{Gr}(\Gamma^{(2)})$ —being pairs of pairs—are unwieldy, but we can identify elements of $\text{Gr}(\Gamma^{(2)})$ with paths of length 3 in (\mathbb{A}, Γ) . Recall from Definition 2.3.3 that the space of paths of length $k - 1$ in (\mathbb{A}, Γ) is

$$\mathbb{A}^{(k)} = \{(x_1, \dots, x_k) \in \mathbb{A}^k \mid \forall 1 \leq i \leq k, x_i \in \Gamma x_{i+1}\}.$$

For $k = 3$ there is a homeomorphism from $\text{Gr}(\Gamma^{(2)})$ to $\mathbb{A}^{(3)}$ which takes $((x_1, x_2), (x_2, x_3)) \in \text{Gr}(\Gamma^{(2)})$ to (x_1, x_2, x_3) . We always identify $\text{Gr}(\Gamma^{(2)})$ with $\mathbb{A}^{(3)}$ in this way.

For each $k \geq 3$ we inductively let $(\text{Gr}(\Gamma^{(k-1)}), \Gamma^{(k)})$ denote the graph system of $(\text{Gr}(\Gamma^{(k-2)}), \Gamma^{(k-1)})$. We inductively obtain homeomorphisms from $\text{Gr}(\Gamma^{(k)})$ to $\mathbb{A}^{(k+1)}$ given by $((x_1, x_2, \dots, x_k), (x_2, x_3, \dots, x_{k+1})) \mapsto (x_1, x_2, \dots, x_{k+1})$. Again we always make this identification. The existence of such homeomorphisms implies that $\mathbb{A}^{(k)}$ is a second-countable compact Hausdorff space for all $k \in \mathbb{N}$.

For each $1 \leq l \leq k$ we let $q_{k,l}: \mathbb{A}^{(k)} \rightarrow \mathbb{A}^{(l)}$ denote the projection

$$q_{k,l}(x_1, \dots, x_k) = (x_1, \dots, x_l).$$

Repeated application of Lemma 2.5.12 implies that $(q_{k,l}, \gamma^{(k)} \mapsto \gamma^{(l)})$ defines a semi-conjugacy. Define $\mu_{l,k}: C(\mathbb{A}^{(l)}) \rightarrow C(\mathbb{A}^{(k)})$ to be the $*$ -homomorphism dual to $q_{k,l}$. That is,

$$\mu_{l,k}(\xi)(x_1, \dots, x_k) = \xi(x_1, \dots, x_l)$$

The Kajiwara-Watatani correspondence associated to $(\mathbb{A}^{(k)}, \Gamma^{(k)})$ is denoted by $(\phi_k, E_{\Gamma^{(k)}})$.

Proposition 2.5.17. *Let (\mathbb{A}, Γ) be an iterated function system and let $1 \leq l \leq k$. Then $(\mu_{l,k}, \mu_{l+1,k+1})$ defines an injective covariant morphism of correspondences from $(\phi_l, E_{\Gamma^{(l)}})$ to $(\phi_k, E_{\Gamma^{(k)}})$. In particular, there is an injective $*$ -homomorphism $\Phi_{l,k}: C^*(\mathbb{A}^{(l)}, \Gamma^{(l)}) \rightarrow C^*(\mathbb{A}^{(k)}, \Gamma^{(k)})$ induced by $(\mu_{l,k}, \mu_{l+1,k+1})$.*

Proof. We just prove the result when $l = 1$ and $k = 2$. The general case follows from the same arguments applied mutatis mutandis.

Since $(q_{2,1}, \gamma^{(2)} \mapsto \gamma)$ is a semiconjugacy, it follows from Proposition 2.2.21 that $(\mu_{1,2}, \mu_{2,3})$ defines an injective morphism of correspondences from (ϕ, E_Γ) to $(\phi_2, E_{\Gamma^{(2)}})$. Lemma 2.5.14 implies that if $(x_1, x_2, x_3) \in B_{\Gamma^{(2)}} \subseteq \mathbb{A}^{(3)}$, then $(x_1, x_2) \in B_\Gamma$. Consequently, $\mu_{1,2}(C_0(\mathbb{A} \setminus p_1(B_\Gamma))) \subseteq C_0(\mathbb{A}^{(2)} \setminus p_1^{(2)}(B_{\Gamma^{(2)}}))$. In particular, Proposition 2.2.19 implies that $\mu_{1,2}(I_{E_\Gamma}) \subseteq I_{E_{\Gamma^{(2)}}}$.

For covariance let $(e_i)_{i \in \mathbb{N}}$ be a frame for E_Γ . Lemma A.3.16 implies that it suffices to check that $(\sum_{i=1}^k \Theta_{\mu_{2,3}(e_i), \mu_{2,3}(e_i)})_{k \in \mathbb{N}}$ is an approximate identity for $\phi_2(\mu_{1,2}(I_{E_\Gamma}))$. To this end, fix $a \in I_E$. For each $\xi \in E_{\Gamma^{(2)}}$, we have $\phi_2(\mu_{1,2}(a))\xi(x) = a(x_1)\xi(x)$. Now fix $\xi \in E_{\Gamma^{(2)}}$ with $\|\xi\| \leq 1$. Compactness of $\mathbb{A}^{(3)}$ implies that for each $k \in \mathbb{N}$ there exists $z \in \mathbb{A}^{(3)}$ (depending on k) which achieves the norm:

$$\begin{aligned} & \left\| \phi_1(a)\xi - \phi(a)_1 \sum_{i=1}^k \mu_{2,3}(e_i) \cdot (\mu_{2,3}(e_i) \mid \xi)_{C(\mathbb{A}^{(2)})} \right\|_\infty \\ &= \left| a(z_1)\xi(z) - \frac{1}{|\Gamma|} \sum_{i=1}^k \sum_{\gamma \in \Gamma} a(z_1) e_i(z_1, z_2) \overline{e_i(\gamma(z_2), z_2)} \xi(\gamma(z_2), z_2, z_3) \right|. \end{aligned}$$

By the Tietze Extension Theorem we can find $f_\xi \in C(\mathbb{A}^{(2)})$ such that $f_\xi(\gamma(z_2), z_2) = \xi(\gamma(z_2), z_2, z_3)$ for all $\gamma \in \Gamma$, and $\|f_\xi\|_\infty \leq \|\xi\|_\infty$. Using the equivalence of norms of Lemma 2.2.8 we see that

$$\begin{aligned}
& \left\| \phi_2(\mu_{1,2}(a))\xi - \phi_2(\mu_{1,2}(a)) \sum_{i=1}^k \Theta_{\mu_{2,3}(e_i), \mu_{2,3}(e_i)} \xi \right\| \\
& \leq \left\| \phi_2(\mu_{1,2}(a))\xi - \phi_2(\mu_{1,2}(a)) \sum_{i=1}^k \mu_{2,3}(e_i) \cdot (\mu_{2,3}(e_i) \mid \xi)_{C(\mathbb{A}^{(2)})} \right\|_{\infty} \\
& = \left| a(z_1) f_{\xi}(z_1, z_2) - \frac{1}{|\Gamma|} \sum_{i=1}^k \sum_{\gamma \in \Gamma} a(z_1) e_i(z_1, z_2) \overline{e_i(\gamma(z_2), z_2)} f_{\xi}(\gamma(z_2), z_2) \right| \\
& \leq \sqrt{|\Gamma|} \left\| \phi_1(a) f_{\xi} - \phi_1(a) \sum_{i=1}^k \Theta_{e_i, e_i} f_{\xi} \right\| \\
& \leq \sqrt{|\Gamma|} \left\| \phi_1(a) - \phi_1(a) \sum_{i=1}^k \Theta_{e_i, e_i} \right\|.
\end{aligned}$$

Proposition A.1.8 implies $(\sum_{i=1}^k \Theta_{e_i, e_i})_{k \in \mathbb{N}}$ is an approximate identity for $\text{End}_A^0(E)$. Consequently, $(\sum_{i=1}^k \Theta_{\mu_{2,3}(e_i), \mu_{2,3}(e_i)})_{k \in \mathbb{N}}$ is an approximate identity for $\phi_2(\mu_{1,2}(I_{E_{\Gamma}}))$, and covariance follows. The final statement follows from Lemma A.3.14. \square

Remark 2.5.18. It is unclear whether $\Phi_{l,k}: C^*(\mathbb{A}^{(l)}, \Gamma^{(l)}) \rightarrow C^*(\mathbb{A}^{(k)}, \Gamma^{(k)})$ from Proposition 2.5.17 is an isomorphism. The reader familiar with directed graph C^* -algebras might suspect that $\Phi_{l,k}$ is an isomorphism, since the graph system $(\text{Gr}(\Gamma), \Gamma^{(2)})$ is analogous to the *dual graph* of a directed graph. For row-finite source free graphs, the C^* -algebra of the dual graph is isomorphic to the C^* -algebra of the original graph [Rae05, Corollary 2.6], and one might expect a similar result here.

2.5.2 Inverse lifted systems

In this section we introduce a way to extend an arbitrary iterated function system to an invertible system via an inverse limit construction. Using this, we show that the Kajiwara-Watatani algebra of an iterated function system always embeds in the Kajiwara-Watatani algebra of an invertible system: the *inverse lifted system*. In Theorem 2.5.32 we show that the Kajiwara-Watatani algebra of the inverse lifted system is isomorphic to a direct limit of Kajiwara-Watatani algebras of graph systems.

To begin, let (\mathbb{A}, Γ) be an iterated function system, and consider the inverse limit

$$\tilde{\mathbb{A}} := \varprojlim (\mathbb{A}^{(k)}, q_{k+1,k}) = \{(x^{(k)})_k \in \prod_k \mathbb{A}^{(k)} \mid x^{(k)} = q_{k+1,k}(x^{(k+1)})\},$$

which we equip with inverse limit topology. That is the weakest topology making the universal projections $q_k: \tilde{\mathbb{A}} \rightarrow \mathbb{A}^{(k)}$ —given by $q_k(x) = (x_1, x_2, \dots, x_k)$ —continuous. We identify $\tilde{\mathbb{A}}$ with the set

$$\{(x_1, x_2, \dots) \in \mathbb{A}^{\mathbb{N}} \mid \forall i \in \mathbb{N}, x_i \in \Gamma x_{i+1}\},$$

and note that the inverse limit topology coincides with the subspace topology inherited from $\mathbb{A}^{\mathbb{N}}$. Since $\tilde{\mathbb{A}}$ can be identified as closed subset of $\mathbb{A}^{\mathbb{N}}$ we see that $\tilde{\mathbb{A}}$ is a second-countable compact Hausdorff space.

For each $\gamma \in \Gamma$ define a map $\gamma^{\mathbb{N}}: \mathbb{A}^{\mathbb{N}} \rightarrow \mathbb{A}^{\mathbb{N}}$ by

$$\gamma^{\mathbb{N}}(x_1, x_2, \dots) = (\gamma(x_1), x_1, x_2, \dots).$$

Each $\gamma^{\mathbb{N}}$ is continuous with respect to the product topology on $\mathbb{A}^{\mathbb{N}}$. Let $\tilde{\gamma}$ denote the restriction of $\gamma^{\mathbb{N}}$ to $\tilde{\mathbb{A}}$ and observe that $\bigcup_{\gamma \in \Gamma} \tilde{\gamma}(\tilde{\mathbb{A}}) = \tilde{\mathbb{A}}$. In particular, $(\tilde{\mathbb{A}}, \{\tilde{\gamma} \mid \gamma \in \Gamma\})$ defines an iterated function system. Following Kieninger, we introduce the inverse lifted system.

Definition 2.5.19 ([Kie02, §5.2]). Let (\mathbb{A}, Γ) be an iterated function system. We call $(\tilde{\mathbb{A}}, \tilde{\Gamma} = \{\tilde{\gamma} \mid \gamma \in \Gamma\})$ the *inverse lifted system* of (\mathbb{A}, Γ) , where $\tilde{\gamma}: \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ is given by

$$\tilde{\gamma}(x_1, x_2, x_3, \dots) = (\gamma(x_1), x_1, x_2, \dots).$$

Remark 2.5.20. The term “lifted” is used because the maps $\tilde{\gamma} \in \tilde{\Gamma}$ are liftings of the corresponding maps $\gamma \in \Gamma$ in the sense that $q_1 \circ \tilde{\gamma} = \gamma \circ q_1$ for each $\gamma \in \Gamma$. Since $q_1: \tilde{\mathbb{A}} \rightarrow \mathbb{A}$ is surjective, the pair $(q_1, \tilde{\gamma} \mapsto \gamma)$ is a semiconjugacy from $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ to (\mathbb{A}, Γ) . Furthermore, $(q_k, \tilde{\gamma} \mapsto \gamma^{(k)})$ is a semiconjugacy from $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ to $(\mathbb{A}^{(k)}, \Gamma^{(k)})$ for all $k \in \mathbb{N}$.

Inverse limit type constructions are frequently used in single function topological dynamics to lift results for homeomorphisms to results for continuous surjections. The main reason we consider the inverse lifted system is because it is always invertible.

Proposition 2.5.21 (cf. [Kie02, Proposition 5.4.12]). *Let (\mathbb{A}, Γ) be an iterated function system. The inverse lifted system $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ is invertible with shift map $\tilde{\sigma}: \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ given by the left shift,*

$$\tilde{\sigma}(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

Proof. Suppose that $\gamma_1, \gamma_2 \in \Gamma$ and that $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ in $\tilde{\mathbb{A}}$ satisfy $\tilde{\gamma}_1(x) = \tilde{\gamma}_2(y)$. Then $(\gamma_1(x_1), x_1, x_2, \dots) = (\gamma_2(y_1), y_1, y_2, \dots)$. It follows that $x = y$, so $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ is invertible. \square

We have the following analogue of Lemma 2.5.12.

Proposition 2.5.22. *If (\mathbb{A}, Γ) is an invertible iterated function system, then $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ is conjugate to (\mathbb{A}, Γ) .*

Proof. As observed in Remark 2.5.20, $(q_1, \tilde{\gamma} \mapsto \gamma)$ is a semiconjugacy. We claim that $q_1: \tilde{\mathbb{A}} \rightarrow \mathbb{A}$ is injective. Suppose that $q_1(x_1, x_2, \dots) = q_1(y_1, y_2, \dots)$. Then $x_1 = y_1$. Let $\sigma: \mathbb{A} \rightarrow \mathbb{A}$ denote the shift map on (\mathbb{A}, Γ) . Since $\sigma \circ \gamma(z) = z$ for all $z \in \mathbb{A}$ and $\gamma \in \Gamma$, we have $x_2 = \sigma(x_1) = y_2$. Continuing inductively, we see that $x_k = \sigma^{k-1}(x_1) = y_k$ for all $k \in \mathbb{N}$. Hence, q_1 is a bijection. Since q_1 is a continuous bijection from a compact space $\tilde{\mathbb{A}}$ to a Hausdorff space \mathbb{A} , it follows that q_1 is a homeomorphism. \square

In light of Proposition 2.5.22 we see that the map $\tilde{\sigma}$ is typically not a local homeomorphism, and $\tilde{\mathbb{A}}$ is typically not totally disconnected. On the other hand, $\tilde{\mathbb{A}}$ is totally disconnected when (\mathbb{A}, Γ) is branch separated (Definition 2.3.1).

Proposition 2.5.23. *If (\mathbb{A}, Γ) is branch separated, and Γ consists of at least two distinct maps, then $\tilde{\mathbb{A}}$ is totally disconnected. If \mathbb{A} has no isolated points, then $\tilde{\mathbb{A}}$ is a Cantor space.*

Proof. Suppose \mathbb{A} is not a singleton and that $x \neq y \in \tilde{\mathbb{A}}$. Let $k \in \mathbb{N}$ be the smallest number such that $x_{k+1} \neq y_{k+1}$. Let $\gamma \in \Gamma$ be such that $x_k = \gamma(x_{k+1})$. Since (\mathbb{A}, Γ) is branch separated $\text{Gr}(\gamma)$ is a clopen neighbourhood of (x_k, x_{k+1}) which does not contain (y_k, y_{k+1}) . Hence, $(\sigma^{k-1})^{-1} \circ q_2^{-1}(x_k, x_{k+1})$ is a clopen neighbourhood of x not containing y . It follows that \mathbb{A} is totally disconnected.

For the second statement, suppose that x is an isolated point of $\tilde{\mathbb{A}}$. Since $\tilde{\mathbb{A}}$ is equipped with the initial topology for the maps $\{q_k\}_{k \in \mathbb{N}}$, there exists $k \in \mathbb{N}$ such that $\{(x_1, \dots, x_k)\}$ is open in $\mathbb{A}^{(k)}$. The projection $(x_1, \dots, x_k) \mapsto x_k$ is the composition of projections $p_2 \circ p_2^{(2)} \circ \dots \circ p_2^{(k-1)}$ each of which is open, so is itself open. Hence, $\{x_k\}$ is open in \mathbb{A} .

Consequently, if \mathbb{A} has no isolated points, then neither does $\tilde{\mathbb{A}}$. Since $\tilde{\mathbb{A}}$ is a non-empty totally disconnected compact metrisable space with no isolated points, it is homeomorphic to a Cantor space. \square

Remark 2.5.24. It might be tempting to think of $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ as a code space for (\mathbb{A}, Γ) when (\mathbb{A}, Γ) is branch separated. However, it is not immediately obvious whether $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ is conjugate to $(\Omega_{|\Gamma|}, \bar{\Gamma})$ under any abstract isomorphism of $\tilde{\mathbb{A}}$ with $\Omega_{|\Gamma|}$ coming from Proposition 2.5.23.

Since $\tilde{\mathbb{A}}$ is defined in terms of an inverse limit, it is not entirely surprising that the iterated function system $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ can be characterised via a universal property.

Theorem 2.5.25. *Let (\mathbb{A}, Γ) be an iterated function system. Then $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ is the smallest invertible lifting of (\mathbb{A}, Γ) in the following sense: if (\mathbb{B}, Λ) is an invertible iterated function system with $|\Lambda| = |\Gamma|$ and $(f, \alpha): (\mathbb{B}, \Lambda) \rightarrow (\mathbb{A}, \Gamma)$ is a semiconjugacy, then there is a unique semiconjugacy $(\tilde{f}, \tilde{\alpha}): (\mathbb{B}, \Lambda) \rightarrow (\tilde{\mathbb{A}}, \tilde{\Gamma})$ that makes the following diagram commute:*

$$\begin{array}{ccc} & (\tilde{\mathbb{A}}, \tilde{\Gamma}) & \\ \tilde{f}, \tilde{\alpha} \nearrow & \downarrow (q_1, \tilde{\gamma} \mapsto \gamma) & \\ (\mathbb{B}, \Lambda) & \xrightarrow{(f, \alpha)} & (\mathbb{A}, \Gamma) \end{array} \quad .$$

Proof. Since (\mathbb{B}, Λ) is invertible, (\mathbb{B}, Λ) is conjugate to $(\tilde{\mathbb{B}}, \tilde{\Lambda})$. Let τ be the shift map for (\mathbb{B}, Λ) . For each $k \in \mathbb{N}$ consider the map $r_k: \mathbb{B} \rightarrow \mathbb{A}^{(k)}$ given by,

$$r_k(y) = (f(y), f \circ \tau(y), \dots, f \circ \tau^{k-1}(y)).$$

This is well-defined since $\tau(y) = \gamma^{-1}(y)$ for some $\gamma \in \Lambda$, and $f \circ \gamma = \alpha(\gamma) \circ f$ for all $\gamma \in \Lambda$. Since $r_k \circ q_{k+1, k} = r_{k+1}$, the universal property of $\tilde{\mathbb{A}}$ as an inverse limit gives a unique continuous map $\tilde{f}: \mathbb{B} \rightarrow \tilde{\mathbb{A}}$ satisfying

$$\tilde{f}(y) = (f(y), f \circ \tau(y), f \circ \tau^2(y), \dots).$$

Define $\tilde{\alpha}: \Lambda \rightarrow \tilde{\Gamma}$ by $\tilde{\alpha}(\gamma) = \widetilde{\alpha(\gamma)}$ for each $\gamma \in \Lambda$. For each $\gamma \in \Lambda$ and $y \in \mathbb{B}$ we have,

$$\begin{aligned}
\tilde{f} \circ \gamma(y) &= (f \circ \gamma(y), f \circ \tau \circ \gamma(y), f \circ \tau^2 \circ \gamma(y), \dots) \\
&= (f \circ \gamma(y), f(y), f \circ \tau(y), \dots) \\
&= (\alpha(\gamma) \circ f(y), f(y), f(\tau(y)), \dots) \\
&= \widetilde{\alpha(\gamma)} \circ \tilde{f}(y) \\
&= \tilde{\alpha}(\gamma) \circ \tilde{f}(y).
\end{aligned}$$

For surjectivity of \tilde{f} we claim that each r_k is surjective. Fix $(x_1, \dots, x_k) \in \mathbb{A}^{(k)}$ with $x_i = \gamma_i(x_{i+1})$. Let $y \in \mathbb{B}$ be such that $x_k = f(y)$. Then,

$$r_k(\tilde{\alpha}^{-1}(\gamma_1) \circ \dots \circ \tilde{\alpha}^{-1}(\gamma_{k-1}) \circ f(y)) = (x_1, x_2, \dots, x_k).$$

Now suppose that $x = (x_1, x_2, \dots) \in \tilde{\mathbb{A}}$. For each $k \in \mathbb{N}$ fix $y^{(k)} \in \mathbb{B}$ such that $r_k(y^{(k)}) = (x_1, \dots, x_k)$. Since \mathbb{B} is compact, we can pass to a convergent subsequence of $(y^{(k)})_{k \in \mathbb{N}}$ with limit $y \in \mathbb{B}$. Continuity of r_k implies that $r_k(y) = (x_1, \dots, x_k)$ for each $k \in \mathbb{N}$. Consequently, $\tilde{f}(y) = x$. Hence, $(\tilde{f}, \tilde{\alpha})$ is a semiconjugacy. \square

Recall that if (\mathbb{A}, Γ) admits a code map $\pi: \Omega_{|\Gamma|} \rightarrow \mathbb{A}$ then $(\pi, \bar{\gamma} \mapsto \gamma)$ is a semi-conjugacy. The following corollary is an immediate consequence of Theorem 2.5.25. In particular, the inverse lifted system $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ acts as an intermediate between (\mathbb{A}, Γ) and the code space $(\Omega_{|\Gamma|}, \bar{\Gamma})$.

Corollary 2.5.26. *If (\mathbb{A}, Γ) admits a code map $\pi: \Omega_{|\Gamma|} \rightarrow \mathbb{A}$ then $\tilde{\pi}: \Omega_{|\Gamma|} \rightarrow \tilde{\mathbb{A}}$ defined by*

$$\tilde{\pi}(w) = (\pi(w), \pi(\sigma(w)), \pi(\sigma^2(w)), \dots)$$

is a code map for $(\tilde{\mathbb{A}}, \tilde{\Gamma})$, and $q_1 \circ \tilde{\pi} = \pi$.

Denote the algebra $C(\tilde{\mathbb{A}})$ of continuous functions on $\tilde{\mathbb{A}}$ by \tilde{A} . The Kajiwara-Watatani correspondence associated to $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ is denoted by $(\tilde{\phi}, E_{\tilde{\Gamma}})$. Since $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ is an invertible system, Proposition 2.5.8 implies that the correspondence $(\tilde{\phi}, E_{\tilde{\Gamma}})$ is isomorphic to $(\text{id}, \tilde{A}_{\tilde{\sigma}})$, and we make this identification.

For each $k \in \mathbb{N}$, let $\mu_k: C(\mathbb{A}^{(k)}) \rightarrow C(\tilde{\mathbb{A}})$ denote the $*$ -homomorphism induced by $q_k: \tilde{\mathbb{A}} \rightarrow \mathbb{A}^{(k)}$. For $f \in C(\mathbb{A}^{(k)})$ we have

$$\mu_k(f)(x) = f(x_1, x_2, \dots, x_k).$$

As each q_k is surjective μ_k is injective. Since $(q_k, \tilde{\gamma} \mapsto \gamma^{(k)})$ is a semiconjugacy from $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ to the graph system $(\mathbb{A}^{(k)}, \Gamma^{(k)})$, it follows from Proposition 2.2.21 that (μ_k, μ_{k+1}) is an injective morphism of correspondences from (ϕ, E_{Γ}) to $(\tilde{\phi}, E_{\tilde{\Gamma}})$. We will show in Proposition 2.5.29 that (μ_k, μ_{k+1}) is covariant. Before we show this, we relate the branched set of $(\mathbb{A}^{(k)}, \Gamma^{(k)})$ to the branched set of $(\tilde{\mathbb{A}}, \tilde{\Gamma})$.

Recall that upon identifying $(\tilde{\phi}, E_{\tilde{\Gamma}})$ with $(\text{id}, \tilde{A}_{\tilde{\sigma}})$ we identify $p_1(B_{\tilde{\Gamma}})$ with $B_{\tilde{\Gamma}}$. In particular,

$$B_{\tilde{\Gamma}} = \{x \in \tilde{\mathbb{A}} \mid \tilde{\sigma} \text{ is not locally injective at } x\}.$$

We have the following analogue of Lemma 2.5.14.

Lemma 2.5.27. *Let (\mathbb{A}, Γ) be an iterated function system with inverse lifted system $(\tilde{\mathbb{A}}, \tilde{\Gamma})$. If $x \in B_{\tilde{\Gamma}}$, then $(x_1, x_2, \dots, x_{k+1}) \in B_{\Gamma^{(k)}}$ for all $k \in \mathbb{N}$. In particular, $\mu_k(I_{E_{\Gamma^{(k)}}}) \subseteq I_{E_{\tilde{\Gamma}}}$. Conversely, for each $(x_1, x_2, \dots, x_{k+1}) \in B_{\Gamma^{(k)}}$ there exists $x \in B_{\tilde{\Gamma}}$ such that $q_{k+1}(x) = (x_1, x_2, \dots, x_{k+1})$.*

Proof. The proof follows almost identical line of reasoning to that of Lemma 2.5.14. By Lemma 2.2.18 it suffices to show that $\tilde{\sigma}: \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ is locally injective at $x = (x_1, x_2, \dots) \in \tilde{\mathbb{A}}$ whenever $p_2^{(k)}: \mathbb{A}^{(k+1)} \rightarrow \mathbb{A}^{(k)}$ is locally injective at $(x_1, x_2, \dots, x_{k+1})$.

So suppose that p_2 is locally injective at $(x_1, x_2, \dots, x_{k+1}) \in \mathbb{A}^{(k+1)}$ and take an open neighbourhood U of $(x_1, x_2, \dots, x_{k+1})$ such that $p_2^{(k)}|_U$ is injective. Consider the open neighbourhood $V := q_{k+1}^{-1}(U)$ of x , and suppose that $y, z \in V$ are such that $\sigma(y) = \sigma(z)$. It follows that $y_i = z_i$ for all $i \geq 2$. Since $(y_1, y_2, \dots, y_{k+1})$ and $(z_1, z_2, \dots, z_{k+1})$ both belong to U and $y_i = z_i$ for $i \geq 2$, we have $y_1 = z_1$. Therefore, $\tilde{\sigma}|_V$ is injective, so $\tilde{\sigma}$ is locally injective at x . It now follows from Proposition 2.2.19 that $\mu_k(I_{E_{\Gamma^{(k)}}}) \subseteq I_{E_{\tilde{\Gamma}}}$.

Now suppose that $(x_1, x_2, \dots, x_{k+1}) \in B_{\Gamma^{(k)}} \subseteq \mathbb{A}^{(k+1)}$. Since $p_k^{(2)}|_U$ is not injective, for any open neighbourhood U of $(x_1, x_2, \dots, x_{k+1})$, there exist sequences

$$(y_{n,1}, y_{n,2}, \dots, y_{n,k+1}) \rightarrow (x_1, x_2, \dots, x_{k+1}) \text{ and } (y'_{n,1}, y_{n,2}, \dots, y_{n,k+1}) \rightarrow (x_1, x_2, \dots, x_{k+1}),$$

in $\mathbb{A}^{(k+1)}$ such that $y_{n,1} \neq y'_{n,1}$ for all $n \in \mathbb{N}$. Using Lemma 2.5.13 we pass to a subsequence of $(y_{n,k+1})$ such that there exists $\gamma \in \Lambda$ with $x_{k+1} \in \gamma(\mathbb{A})$ and $y_{n,k+1} \in \gamma(\mathbb{A})$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, fix $z_n = (z_{n,1}, z_{n,2}, \dots) \in \tilde{\mathbb{A}}$ such that $z_{n,1} \in \tilde{\gamma}^{-1}(y_{n,k+1})$. Since $\tilde{\mathbb{A}}$ is compact, we can pass to a convergent subsequence of $(z_n)_{n=1}^\infty$ whose limit we denote by $z = (z_1, z_2, \dots)$. As γ is continuous and $y_{n,k+1} \rightarrow x_{k+1}$ we have $\gamma(z) = x_{k+1}$.

Now consider $x := (x_1, x_2, \dots, x_{k+1}, z_1, z_2, \dots) \in \tilde{\mathbb{A}}$. For each open neighbourhood U of $(x_1, x_2, \dots, x_{k+1})$ and each open neighbourhood V of z in $\tilde{\mathbb{A}}$, let

$$W_{U,V} := q_{k+1}^{-1}(U) \cap \tilde{\sigma}^{-(k+1)}(V).$$

Then the $W_{U,V}$ are a neighbourhood base of x . Fix such U and V , and take $N \in \mathbb{N}$ such that $(y_{N,1}, y_{N,2}, \dots, y_{N,k+1}) \in U$, $(y'_{N,1}, y_{N,2}, \dots, y_{N,k+1}) \in U$, and $z_N \in V$. Then $(y_{N,1}, y_{N,2}, \dots, y_{N,k+1}, z_{N,1}, z_{N,2}, \dots)$ and $(y'_{N,1}, y_{N,2}, \dots, y_{N,k+1}, z_{N,1}, z_{N,2}, \dots)$ both belong to $W_{U,V}$. Thus, $\tilde{\sigma}$ is not locally injective at x . Hence, $x \in B_{\tilde{\Gamma}}$. \square

We can modify the arguments of Example 2.5.15 to see that for $(x, y) \in \text{Gr}(\tilde{\Gamma})$ with $(x_1, y_1) \in B_\Gamma$, we do not necessarily have $(x, y) \in B_{\tilde{\Gamma}}$.

Example 2.5.28. Let $([0, 1], \Gamma = \{\gamma_1, \gamma_2, \gamma_3\})$ be the iterated function system of Example 2.5.15. As before, let U be an open neighbourhood of $(1/2, 0)$ in $\text{Gr}(\Gamma)$ contained in $\text{Gr}(\gamma_2)$ as pictured in Figure 2.5. Let $V = \{(\gamma_1(x_1), x_1) \mid (x_1, x_2) \in U\}$. Then V is not open in $\text{Gr}(\Gamma)$. Let $p_1^{(3)}: \mathbb{A}^{(3)} \rightarrow \mathbb{A}^{(2)}$ and $p_2^{(3)}: \mathbb{A}^{(3)} \rightarrow \mathbb{A}^{(2)}$ be the projections $p_1^{(3)}(x_1, x_2, x_3) = (x_1, x_2)$ and $p_2^{(3)}(x_1, x_2, x_3) = (x_2, x_3)$. Then,

$$\begin{aligned} W &:= (p_1^{(3)})^{-1}(\text{Gr}(\gamma_1) \cup \text{Gr}(\gamma_3)) \cap (p_2^{(3)})^{-1}(U) \\ &= \{(x_1, x_2, x_3) \in \mathbb{A}^{(3)} \mid (x_1, x_2) \in V, (x_2, x_3) \in U\} \\ &= \{(\gamma_1(x_1), x_1, x_2) \in \mathbb{A}^{(3)} \mid (x_1, x_2) \in U\} \end{aligned}$$

is an open subset of $\mathbb{A}^{(3)}$ containing $(1/4, 1/2, 0)$. Let $z = (1/4, 1/2, 0, 0, 0, \dots) \in \tilde{\mathbb{A}}$. Then $q_3^{-1}(W)$ is an open neighbourhood of z . Suppose that $y, y' \in q_3^{-1}(W)$ are such that $\sigma(y) = \sigma(y')$. Then $y_2 = y'_2$. Since (y_1, y_2) and (y'_1, y'_2) both belong to V , it follows that $y_1 = \gamma_1(y_2) = y'_1$. As such, $y = y'$. Consequently, σ is injective when restricted to $q_3^{-1}(W)$. Hence, $z \notin B_{\tilde{\Gamma}}$ while $(z_1, z_2) = (1/4, 1/2) \in B_{\Gamma}$. \triangle

Every Kajiwara-Watatani algebra embeds into the algebra of its inverse lifted system.

Proposition 2.5.29. *Let (\mathbb{A}, Γ) be an iterated function system. Then for each $k \in \mathbb{N}$ the pair (μ_k, μ_{k+1}) defines an injective and covariant morphism of correspondences from $(\phi_k, E_{\Gamma^{(k)}})$ to $(\tilde{\phi}, E_{\tilde{\Gamma}})$ which satisfies $\mu_k(I_{E_{\Gamma}}) \subseteq I_{E_{\tilde{\Gamma}}}$. Moreover, (μ_k, μ_{k+1}) induces an injective $*$ -homomorphism $\Phi_k: C^*(\mathbb{A}^{(k)}, \Gamma^{(k)}) \rightarrow C^*(\tilde{\mathbb{A}}, \tilde{\Gamma})$.*

Proof. The proof is almost identical to the proof of Proposition 2.5.17, but we include it for completeness. It suffices to prove the result in the case where $k = 1$ since $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ is conjugate to $(\tilde{\mathbb{A}}^{(k)}, \tilde{\Gamma}^{(k)})$. Since $(q_1, \tilde{\gamma} \mapsto \gamma)$ is a semiconjugacy from $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ to (\mathbb{A}, Γ) , it follows from Proposition 2.2.21 that (μ_1, μ_2) is an injective morphism of correspondences from (ϕ, E_{Γ}) to $(\tilde{\phi}, E_{\tilde{\Gamma}})$. Lemma 2.5.27 implies that $\mu_1(I_{E_{\Gamma}}) \subseteq I_{E_{\tilde{\Gamma}}}$. All that remains to prove is covariance.

For covariance let $(e_i)_{i \in \mathbb{N}}$ be a frame for E_{Γ} . Lemma A.3.16 implies that it suffices to check that $(\sum_{i=1}^k \Theta_{\mu_2(e_i), \mu_2(e_i)})_{k \in \mathbb{N}}$ is an approximate identity for $\tilde{\phi}(\mu_1(I_{E_{\Gamma}}))$. To this end, fix $a \in I_E$. For each $\xi \in E_{\tilde{\Gamma}}$, we have $\tilde{\phi}(\mu_1(a))\xi(x) = a(x_1)\xi(x)$.

Now fix $\xi \in E_{\tilde{\Gamma}}$ with $\|\xi\| \leq 1$. Compactness of $\tilde{\mathbb{A}}$ implies that there exists $z \in \tilde{\mathbb{A}}$ (depending on k) which achieves the norm:

$$\begin{aligned} & \left\| \tilde{\phi}(\mu_1(a))\xi - \tilde{\phi}(\mu_1(a)) \sum_{i=1}^k \mu_2(e_i) \cdot (\mu_2(e_i) \mid \xi)_{\tilde{A}} \right\|_{\infty} \\ &= \left| a(z_1)\xi(z) - \frac{1}{|\Gamma|} \sum_{i=1}^k \sum_{\gamma \in \Gamma} a(z_1)e_i(z_1, z_2) \overline{e_i(\gamma(z_2), z_2)} \xi(\gamma(z_2), z_2, z_3, \dots) \right|. \end{aligned}$$

The Tietze extension theorem yields $f_{\xi} \in C(\text{Gr}(\Gamma))$ such that $\|f_{\xi}\|_{\infty} \leq \|\xi\|_{\infty}$ and $f_{\xi}(\gamma(z_2), z_2) = \xi(\gamma(z_2), z_2, z_3, \dots)$ for all $\gamma \in \Gamma$. Using the equivalence of norms from Lemma 2.2.8 we see that,

$$\begin{aligned} & \left\| \tilde{\phi}(\mu_1(a))\xi - \tilde{\phi}(\mu_1(a)) \sum_{i=1}^k \Theta_{\mu_2(e_i), \mu_2(e_i)} \xi \right\| \\ & \leq \left\| \tilde{\phi}(\mu_1(a))\xi - \tilde{\phi}(\mu_1(a)) \sum_{i=1}^k \mu_2(e_i) \cdot (\mu_2(e_i) \mid \xi)_{\tilde{A}} \right\|_{\infty} \\ &= \left| a(z_1)f_{\xi}(z_1, z_2) - \frac{1}{|\Gamma|} \sum_{i=1}^k \sum_{\gamma \in \Gamma} a(z_1)e_i(z_1, z_2) \overline{e_i(\gamma(z_2), z_2)} f_{\xi}(\gamma(z_2), z_2) \right| \\ & \leq \left\| \phi(a)f_{\xi} - \phi(a) \sum_{i=1}^k \Theta_{e_i, e_i} f_{\xi} \right\|_{\infty} \\ & \leq \sqrt{|\Gamma|} \left\| \phi(a)f_{\xi} - \phi(a) \sum_{i=1}^k \Theta_{e_i, e_i} f_{\xi} \right\| \end{aligned}$$

$$\leq \sqrt{|\Gamma|} \left\| \phi(a) - \sum_{i=1}^k \Theta_{e_i, e_i} \right\|.$$

Proposition A.1.8 implies $(\sum_{i=1}^k \Theta_{e_i, e_i})_{k \in \mathbb{N}}$ is an approximate identity for $\text{End}_A^0(E)$. Consequently, $(\sum_{i=1}^k \Theta_{\mu_2(e_i), \mu_2(e_i)})_{k \in \mathbb{N}}$ is an approximate identity for $\tilde{\phi}(\mu_2(I_{E_\Gamma}))$, and covariance follows. The final statement follows from Lemma A.3.14. \square

Remark 2.5.30. Just as with Proposition 2.5.17, it is unclear whether the map Φ_k from Proposition 2.5.29 is an isomorphism.

We complete this section by showing that $C^*(\tilde{\mathbb{A}}, \tilde{\Gamma})$ can be realised as the direct limit of the algebras $C^*(\mathbb{A}^{(k)}, \Gamma^{(k)})$. First, we require the following Lemma.

Lemma 2.5.31. *Let (\mathbb{A}, Γ) be an iterated function system. Then*

$$I_{\tilde{\Gamma}} \cong \varinjlim (I_{\Gamma^{(k)}}, \mu_{k, k+1}).$$

Proof. Since $\tilde{\mathbb{A}} \simeq \varinjlim (\mathbb{A}^{(k)}, q_{k, k+1})$, we have $C(\tilde{\mathbb{A}}) \cong \varinjlim (C(\mathbb{A}^{(k)}), \mu_{k, k+1})$. Lemma 2.5.14 implies that $\mu_{k, k+1}(I_{\Gamma^{(k)}}) \subseteq I_{\Gamma^{(k+1)}}$. Fix $a \in I_{\tilde{\Gamma}}$ and $\varepsilon > 0$. We claim that there exists $k \in \mathbb{N}$ and $a_k \in I_{\Gamma^{(k)}}$ such that $\|a - \mu_k(a_k)\| < \varepsilon$ in $C(\tilde{\mathbb{A}})$. Let $k \in \mathbb{N}$ and $b_k \in C(\tilde{\mathbb{A}}^{(k)})$ be such that $\|a - \mu_k(b_k)\| < \varepsilon/2$. Then we have $|b_k(x_1, x_2, \dots, x_k)| < \varepsilon/2$ for all $x = (x_1, x_2, \dots) \in B_{\tilde{\Gamma}}$ by Lemma 2.5.27. Lemma 2.5.27 gives $|b_k(x_1, x_2, \dots, x_k)| < \varepsilon$ for all $(x_1, x_2, \dots, x_k) \in B_{\Gamma^{(k-1)}} = p_1^{(k)}(B_{\Gamma^{(k)}})$. Let $q_k: A^{(k)} \rightarrow A^{(k)}/I_{\Gamma^{(k)}} \cong C_0(B_{\Gamma^{(k-1)}})$ denote the quotient map, which coincides with the restriction map. Then $\|q_k(b_k)\| = \inf\{\|b_k - c\| : c \in I_{\Gamma^{(k)}}\} < \varepsilon/2$. Hence, there exists $a_k \in I_{\Gamma^{(k)}}$ such that $\|b_k - a_k\| < \varepsilon/2$. Thus, $\|\mu_k(a_k) - a\| < \varepsilon$.

To see that $I_{\tilde{\Gamma}}$ satisfies the universal property for direct limits, let $\Theta_k: I_{\Gamma^{(k)}} \rightarrow B$ be $*$ -homomorphisms into a C^* -algebra B satisfying $\Theta_k = \Theta_{k+1} \circ \mu_{k, k+1}$ for all $k \in \mathbb{N}$. Let $a \in I_{\tilde{\Gamma}}$ and for each $k \in \mathbb{N}$ take $a_k \in I_{\Gamma^{(k)}}$ such that $a = \lim_k \mu_k(a_k)$. Then $(\Theta_k(a_k))_{k=1}^\infty$ is Cauchy and setting $\Theta(a) = \lim_k \Theta_k(a_k)$ yields a well-defined $*$ -homomorphism $\psi: I_{\tilde{\Gamma}} \rightarrow B$ satisfying $\Theta \circ \mu_k = \Theta_k$. \square

Theorem 2.5.32. *Let (\mathbb{A}, Γ) be an iterated function system. Then*

$$C^*(\tilde{\mathbb{A}}, \tilde{\Gamma}) \cong \varinjlim (C^*(\mathbb{A}^{(k)}, \Gamma^{(k)}), \Phi_{k, k+1}).$$

Proof. Let $L := \varinjlim (C^*(\mathbb{A}^{(k)}, \Gamma^{(k)}), \Phi_{k, k+1})$. Let $\iota_k: C^*(\mathbb{A}^{(k)}, \Gamma^{(k)}) \rightarrow L$ denote the universal inclusions. We first claim that there is a covariant representation (π, ψ) of $(\tilde{\phi}, E_{\tilde{\Gamma}})$ in L . Since $\tilde{\mathbb{A}} \simeq \varinjlim (\mathbb{A}^{(k)}, q_{k, k+1})$, we have $C(\tilde{\mathbb{A}}) \cong \varinjlim (C(\mathbb{A}^{(k)}), \mu_{k, k+1})$. Since $i_{A^{(k)}}: A^{(k)} \rightarrow C^*(\mathbb{A}^{(k)}, \Gamma^{(k)})$ is a $*$ -homomorphism satisfying $i_{A^{(k+1)}} \circ \mu_{k, k+1} = \Phi_{k, k+1} \circ i_{A^{(k)}}$ it follows from the universal property of direct limits that there is a unique $*$ -homomorphism $\pi: C(\tilde{\mathbb{A}}) \rightarrow L$ satisfying $\pi \circ \mu_k = \iota_k \circ i_{A^{(k)}}$.

To construct ψ , note that the equivalence of norms from Lemma 2.2.8 implies that

for each $\xi \in E_{\tilde{\Gamma}}$ and $k \in \mathbb{N}$ there exists $\xi_k \in E_{\Gamma(k)}$ such that $\lim_k \mu_{k+1}(\xi_k) = \xi$. For $k \geq m$,

$$\begin{aligned} \|\iota_k \circ i_{E_{\Gamma(k)}}(\xi_k) - \iota_m \circ i_{E_{\Gamma(m)}}(\xi_m)\| &= \|\iota_k \circ i_{E_{\Gamma(k)}}(\xi_k) - \iota_k \circ \Phi_{m,k} \circ i_{E_{\Gamma(m)}}(\xi_m)\| \\ &\leq \|i_{E_{\Gamma(k)}}(\xi_k) - i_{E_{\Gamma(k)}} \circ \mu_{m,k}(\xi_m)\| \\ &= \|\xi_k - \mu_{m,k}(\xi_m)\| \\ &= \|\mu_k(\xi_k) - \mu_m(\xi_m)\|. \end{aligned}$$

It follows that $(\iota_k \circ i_{E_{\Gamma(k)}}(\xi_k))_{k \in \mathbb{N}}$ is Cauchy in L . Set $\psi(\xi) = \lim_k \iota_k \circ i_{E_{\Gamma(k)}}(\xi_k)$. A similar calculation to the above shows that ψ extends to a well-defined linear map $\psi: E_{\tilde{\Gamma}} \rightarrow L$, satisfying $\psi \circ \mu_{k+1} = \iota_k \circ i_{E_{\Gamma(k)}}$.

We now check that $\psi(\xi)\pi(a) = \psi(\xi \cdot a)$, that $\pi(a)\psi(\xi) = \psi(\tilde{\phi}(a)\xi)$, and that $\pi((\xi | \eta)_{\tilde{A}}) = \psi(\xi)^*\psi(\eta)$ for all $a \in \tilde{A}$ and $\xi, \eta \in E_{\tilde{\Gamma}}$. We compute one of these equalities, the other two follow from a similar calculation. Fix $\xi, \eta \in E_{\tilde{\Gamma}}$ and suppose that $\xi = \lim_k \mu_{k+1}(\xi_k)$ and $\eta = \lim_k \mu_{k+1}(\eta_k)$. Then,

$$\psi(\xi)^*\psi(\eta) = \lim_k \iota_k(i_{E_{\Gamma(k)}}(\xi_k)^*i_{E_{\Gamma(k)}}(\eta_k)) = \lim_k \iota_k \circ i_{A(k)}((\xi_k | \eta_k)_{A(k)}).$$

It therefore suffices to show that $\mu_k((\xi_k | \eta_k)_{A(k)}) \rightarrow (\xi | \eta)_{\tilde{A}}$. To this end, we compute

$$\|\mu_k((\xi_k | \eta_k)_{A(k)}) - (\xi | \eta)_{\tilde{A}}\| \leq \max_{\gamma \in \Gamma} \sup_{x \in \tilde{A}} |\mu_{k+1}(\xi_k)(\tilde{\gamma}(x)) \overline{\mu_{k+1}(\eta_k)(\tilde{\gamma}(x))} - \xi(\tilde{\gamma}(x)) \overline{\eta(\tilde{\gamma}(x))}|.$$

Since $\mu_{k+1}(\xi_k) \rightarrow \xi$ and $\mu_{k+1}(\eta_k) \rightarrow \eta$, it follows from continuity of multiplication in \tilde{A} that $\mu_k((\xi_k | \eta_k)_{A(k)}) \rightarrow (\xi | \eta)_{\tilde{A}}$. Thus, $\pi((\xi | \eta)_{\tilde{A}}) = \psi(\xi)^*\psi(\eta)$.

For covariance, fix $a \in I_{\tilde{\Gamma}}$. We show that $\pi(a) = \psi^{(1)}(\tilde{\phi}(a))$. By Lemma 2.5.31, for each $k \in \mathbb{N}$ there exists $a_k \in I_{\Gamma(k)}$ such that $a = \lim_k \mu_k(a_k)$. Covariance of $(i_{A(k)}, i_{E_{\Gamma(k)}})$ implies that

$$\pi(a) = \lim_k \iota_k \circ i_{A(k)}(a_k) = \lim_k \iota_k \circ i_{E_{\Gamma(k)}}^{(1)} \circ \phi_k(a_k).$$

Since $\psi \circ \mu_{k+1} = \iota_k \circ i_{E_{\Gamma(k)}}$, for each $\xi_k, \eta_k \in E_{\Gamma(k)}$, we have $\iota_k \circ i_{E_{\Gamma(k)}}^{(1)}(\Theta_{\xi_k, \eta_k}) = \psi^{(1)} \circ \mu_{k+1}^{(1)}(\Theta_{\xi_k, \eta_k})$. It follows that $\iota_k \circ i_{E_{\Gamma(k)}}^{(1)} = \psi^{(1)} \circ \mu_{k+1}^{(1)}$. Since (μ_k, μ_{k+1}) is covariant, $\mu_{k+1}^{(1)} \circ \phi_k(a_k) = \tilde{\phi} \circ \mu_k(a_k)$. It now follows that

$$\pi(a) = \lim_k \psi^{(1)} \circ \tilde{\phi} \circ \mu_k(a_k) = \psi^{(1)}(\tilde{\phi}(a)).$$

Hence, (π, ψ) is covariant. The universal property of Cuntz-Pimsner algebras now implies that (π, ψ) induces a $*$ -homomorphism $\Psi: C^*(\tilde{A}, \tilde{\Gamma}) \rightarrow L$ satisfying $\Psi \circ i_{\tilde{A}} = \pi$ and $\Psi \circ i_{E_{\tilde{\Gamma}}} = \psi$.

Now fix a covariant representation (α, β) of $(\tilde{\phi}, E_{\tilde{\Gamma}})$ in a C^* -algebra B . We aim to show that there is a unique $*$ -homomorphism $\Theta: L \rightarrow B$ such that $\Theta \circ \pi = \alpha$ and $\Theta \circ \psi = \beta$. Once this is shown, the statement of the proposition follows from the uniqueness of Cuntz-Pimsner algebras with regard to their universal property.

Since $(\mu_k, \mu_{k+1}): (\phi_k, E_{\Gamma(k)}) \rightarrow C^*(\tilde{A}, \tilde{\Gamma})$ is a covariant morphism, the composition $(\alpha \circ \mu_k, \beta \circ \mu_{k+1})$ is a covariant representation of $(\phi_k, E_{\Gamma(k)})$ in B . The universal property

of $C^*(\mathbb{A}^{(k)}, \Gamma^{(k)})$ induces a unique $*$ -homomorphism $\Theta_k: C^*(\mathbb{A}^{(k)}, \Gamma^{(k)}) \rightarrow B$ such that $\Theta_k \circ i_{A^{(k)}} = \alpha \circ \mu_k$ and $\Theta_k \circ i_{E_{\Gamma^{(k)}}} = \beta \circ \mu_{k+1}$. Now,

$$\Theta_{k+1} \circ \Phi_{k,k+1} \circ i_{A^{(k)}} = \Theta_{k+1} \circ i_{A^{(k+1)}} \circ \mu_{k,k+1} = \alpha \circ \mu_{k+1} \circ \mu_{k,k+1} = \alpha \circ \mu_k = \Theta_k \circ i_{A^{(k)}}$$

and similarly $\Theta_{k+1} \circ \Phi_{k,k+1} \circ i_{E_{\Gamma^{(k)}}} = \Theta_k \circ i_{E_{\Gamma^{(k)}}}$. Since $C^*(\mathbb{A}^{(k)}, \Gamma^{(k)})$ is generated by the image of $i_{A^{(k)}}$ and $i_{E_{\Gamma^{(k)}}}$ it follows that $\Theta_{k+1} \circ \Phi_{k,k+1} = \Theta_k$. Hence, the universal property of L as a direct limit induces a unique $*$ -homomorphism $\Theta: L \rightarrow B$ satisfying $\Theta_k = \Theta \circ \iota_k$ for each $k \in \mathbb{N}$. It is now straightforward to check that $\Theta \circ \pi = \alpha$ and $\Theta \circ \psi = \beta$.

Since L is universal for covariant representations of $(\tilde{\phi}, E_{\tilde{\Gamma}})$, the uniqueness of Cuntz-Pimsner algebras implies that $L \cong C^*(\tilde{\mathbb{A}}, \tilde{\Gamma})$. \square

2.6 | A Markov operator defining E_{Γ}

In this section we realise the correspondence E_{Γ} as the KSGNS space of a positive map $\mathcal{L}: C(\mathbb{A}) \rightarrow C(\mathbb{A})$. This was explored in the contractive case in [IMV12] where \mathcal{L} is referred to as a *Markov operator*, however there is essentially no difference in the situation of general iterated function systems. Since E_{Γ} arises from a KSGNS construction, we get a direct sum decomposition of E_{Γ} which we exploit in Section 2.7 to build a frame for E_{Γ} . Finally, we show that for an invertible system $C^*(\mathbb{A}, \Gamma)$ is isomorphic to an Exel crossed product.

To begin, we have the following result which relates iterated function systems to Markov operators.

Proposition 2.6.1. *Let \mathbb{A} be a second-countable compact Hausdorff space and let $\Gamma = \{\gamma_1, \dots, \gamma_N\}$ be a collection of continuous self-mappings on \mathbb{A} . Consider the unital positive mapping $\mathcal{L}: C(\mathbb{A}) \rightarrow C(\mathbb{A})$ given by*

$$\mathcal{L}(a)(x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} a(\gamma(x)). \quad (2.11)$$

*Then \mathcal{L} is faithful—in the sense that $\mathcal{L}(a^*a) = 0$ implies $a = 0$ —if and only if $\mathbb{A} = \bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A})$. That is, \mathcal{L} is faithful if and only if (\mathbb{A}, Γ) is an iterated function system.*

Proof. First suppose that (\mathbb{A}, Γ) is an iterated function system. If $a \in C(\mathbb{A})$ and $\mathcal{L}(a^*a) = 0$ then $|a(\gamma(x))|^2 = 0$ for all $x \in \mathbb{A}$ and $\gamma \in \Gamma$. Since $\mathbb{A} = \bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A})$ it follows that $a = 0$.

Now suppose that \mathcal{L} is faithful and fix $x \in \mathbb{A}$. Let $(U_n)_{n=1}^{\infty}$ be a countable neighbourhood base of x . Urysohn's Lemma and normality of \mathbb{A} gives a sequence of positive functions $a_n \in C(\mathbb{A})$ such that $\text{supp}(a_n) \subseteq U_n$ and $a_n(x) = 1$. Since $a_n \neq 0$, faithfulness of \mathcal{L} implies $\mathcal{L}(a_n)(y_n) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} a_n(\gamma(y_n))$ is non-zero for some $y_n \in \mathbb{A}$. In particular, there exists $\gamma_n \in \Gamma$ such that $a_n(\gamma_n(y_n))$ is non-zero. The support condition on a_n implies that $\gamma_n(y_n) \rightarrow x$. By passing to a subsequence we can assume that $\gamma_n = \gamma_{n+1}$ for all $n \in \mathbb{N}$, call this map γ' . By compactness of \mathbb{A} we can pass to a convergence subsequence of $(y_n)_{n=1}^{\infty}$ which we assume converges to $y \in \mathbb{A}$. It now follows from continuity of γ' that $\gamma'(y) = x$. In particular $x \in \bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A})$. \square

Let (\mathbb{A}, Γ) be an iterated function system. Both the graph of Γ and the branched

index can be recovered from \mathcal{L} . For the graph note that

$$\text{Gr}(\Gamma) = \{(x, y) \in \mathbb{A} \times \mathbb{A} \mid \text{for all positive } a \in A, \mathcal{L}(a)(y) = 0 \implies a(x) = 0\}.$$

For the branched index, fix $(\gamma(y), y) \in \text{Gr}(\Gamma)$. By Urysohn's Lemma there exists $a \in C(\mathbb{A})$ such that $a(\gamma(y)) = 1$ and $a(\gamma'(y)) = 0$ for all $\gamma' \in \Gamma$ with $\gamma(y) \neq \gamma'(y)$. Then,

$$b(\gamma(y), y) = |\Gamma| \mathcal{L}(a)(y).$$

Every strict completely positive mapping Φ between C^* -algebras can be represented as the compression of a $*$ -homomorphism [Lan95, Theorem 5.6]. This result is known as the KSGNS construction, and part of the construction involves building a correspondence which we now describe in our setting.

Definition 2.6.2 ([Lan95, Theorem 5.6]). Let X be a compact Hausdorff space and let $\Phi: C(X) \rightarrow C(X)$ be a positive linear map such that $\Phi(1) = 1$. The *KSGNS-correspondence* F_Φ associated to Φ is the completion of the algebraic tensor product $C(X) \odot C(X)$ with respect to the seminorm induced by the A -valued sesquilinear form

$$(a_1 \odot b_1 \mid a_2 \odot b_2) = a_1^* \Phi(b_1^* b_2) a_2.$$

We write $a \otimes b$ for the image of $a \odot b$ in the completion. The left and right actions of $C(X)$ on F_Φ satisfy $c \cdot (a \otimes b) \cdot d = (ca) \otimes b$ and $(a \otimes b) \cdot c = a \otimes (bc)$ for all $a, b, c \in C(X)$.

Proposition 2.6.3 ([IMV12, Proposition 2.2]). Let (\mathbb{A}, Γ) be an iterated function system. Then the Kajiwara-Watatani correspondence E_Γ is isomorphic to the KSGNS correspondence $F_\mathcal{L}$ for the positive map \mathcal{L} given in (2.11).

Proof. Define $\psi: A \odot A \rightarrow E_\Gamma$ by

$$\psi(a \odot b)(x, y) = a(x)b(y), \quad a \odot b \in A \odot A.$$

For each $a_1, a_2, b_1, b_2 \in A$ we have,

$$\begin{aligned} (\psi(a_1 \odot b_1) \mid \psi(a_2 \odot b_2))_A(x) &= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{a_1(\gamma(x)) a_2(x)} b_1(\gamma(x)) b_2(x) \\ &= (a_2 \mathcal{L}(a_1^* b_1) b_2)(x) \\ &= (a_1 \odot b_1 \mid a_2 \odot b_2)_\mathcal{L}(x) \end{aligned}$$

It follows that ψ extends to an isometric linear map $\overline{\psi}: F_\Phi(\mathcal{L}) \rightarrow E_\Gamma$. It is also straightforward to check that $\psi((a \odot b) \cdot c) = \psi(a \odot b) \cdot c$ and $\psi(c \cdot (a \odot b)) = c \cdot \psi(a \odot b)$ for all $a, b, c \in A$. Consequently, $(\text{id}, \overline{\psi})$ defines a morphism of correspondences. An application of the Stone-Weierstrass theorem implies that $\overline{\psi}$ has dense range, giving surjectivity. \square

Since E_Γ can be realised as the KSGNS correspondence for \mathcal{L} , it follows from [Lan95, Theorem 5.6] that there is an operator $V_\mathcal{L} \in \text{Hom}_A(A, E_\Gamma)$ which conjugates $\phi: A \rightarrow \text{End}_A(E_\Gamma)$ to $\mathcal{L}: A \rightarrow A$, in the sense that

$$\mathcal{L}(a)b = V_\mathcal{L}^* \phi(a) V_\mathcal{L} b \tag{2.12}$$

for all $b \in B$ (cf. [Lan95, Theorem 5.6]). The operator $V_{\mathcal{L}}$ together with its adjoint can be described explicitly. Indeed, it is straightforward to check that the operators $V_{\mathcal{L}} \in \text{Hom}_A(A, E_{\Gamma})$ and $V_{\mathcal{L}}^* \in \text{Hom}_A(E_{\Gamma}, A)$ defined by

$$(V_{\mathcal{L}}b)(x, y) = b(y) \quad \text{and} \quad (V_{\mathcal{L}}^*\xi)(x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \xi(\gamma(x), x)$$

for $b \in A$ and $\xi \in E_{\Gamma}$, are mutually adjoint and satisfy (2.12). Moreover, $V_{\mathcal{L}}$ is an isometry with $V_{\mathcal{L}}^*V_{\mathcal{L}} = \text{id}_A$ and $V_{\mathcal{L}}V_{\mathcal{L}}^* = P_{\Gamma}$, where $P_{\Gamma} \in \text{End}_A(E_{\Gamma})$ is the projection given by

$$P_{\Gamma}(\xi)(x, y) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \xi(\gamma(y), y). \quad (2.13)$$

The existence of the projection P_{Γ} implies that as a right A -module E_{Γ} admits a direct sum decomposition.

Proposition 2.6.4. *Let*

$$E_{\Gamma}^0 := \{\xi \in C(\text{Gr}(\Gamma)) \mid \sum_{\gamma \in \Gamma} \xi(\gamma(x), x) = 0 \text{ for all } x \in \mathbb{A}\}$$

regarded as an A -submodule of E_{Γ} . Let $p_2^: A \rightarrow E_{\Gamma}$ be the $*$ -homomorphism dual to the projection $p_2: \text{Gr}(\Gamma) \rightarrow \mathbb{A}$ onto the second factor. Then $p_2^*(A) \cong A_A$, and as a right Hilbert A -module E_{Γ} decomposes as a direct sum*

$$E_{\Gamma} = p_2^*(A) \oplus E_{\Gamma}^0.$$

Proof. Using the projection P_{Γ} of (2.13) we have a decomposition of right Hilbert A -modules $E_{\Gamma} = P_{\Gamma}E_{\Gamma} \oplus (1 - P_{\Gamma})E_{\Gamma}$. If $\xi \in P_{\Gamma}E_{\Gamma}$, then

$$\xi(x, y) = P_{\Gamma}\xi(x, y) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \xi(\gamma(y), y)$$

for all $(x, y) \in \text{Gr}(\Gamma)$. It follows that $P_{\Gamma}E_{\Gamma} = \{\xi \in E_{\Gamma} \mid \xi(x, y) = \xi(x', y) \text{ for all } y \in \mathbb{A} \text{ and } x, x' \in \Gamma y\}$. In particular, $P_{\Gamma}E_{\Gamma} = p_2^*(A)$.

If $\xi \in (1 - P_{\Gamma})E_{\Gamma}$, then

$$\xi(x, y) = (1 - P_{\Gamma})\xi(x, y) = \xi(x, y) - \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \xi(\gamma(y), y)$$

for all $(x, y) \in \text{Gr}(\Gamma)$, forcing $\sum_{\gamma \in \Gamma} \xi(\gamma(y), y) = 0$. Consequently, $(1 - P_{\Gamma})E_{\Gamma} = E_{\Gamma}^0$.

Finally, since p_2 is surjective, p_2^* is injective, so $p_2^*(A) \cong A_A$. \square

The direct sum decomposition of Proposition 2.6.4 does not typically respect the left action of A . In particular, the direct sum is not a direct sum of C^* -correspondences.

In [Kwa17] Kwaśniewski introduced the notion of crossed products by completely positive maps as a far-reaching generalisation of both Exel crossed products [Exe03] and graph C^* -algebras. We use the following definition.

Definition 2.6.5 ([Kwa17]). Let A be a C^* -algebra and let $\Phi: A \rightarrow A$ be a strict completely positive map. The *crossed product* $C^*(A, \Phi)$ of A by Φ is the Cuntz-Pimsner algebra \mathcal{O}_{F_Φ} of the KSGNS correspondence F_Φ .

In its original formulation $C^*(A, \Phi)$ was defined as the quotient of the universal C^* -algebra generated by representations of (A, Φ) , by so called redundancies (based on an earlier concept of Exel [Exe03]). The definition given above is shown in [Kwa17, Theorem 3.13] to agree with the formulation in terms of redundancies. Proposition 2.6.3 now immediately implies the following.

Proposition 2.6.6. *Let (i_A, i_{E_Γ}) be the universal representation of (ϕ, E_Γ) in $C^*(\mathbb{A}, \Gamma)$. The Kajiwara-Watatani algebra $C^*(\mathbb{A}, \Gamma)$ is isomorphic to the crossed product $C^*(C(\mathbb{A}), \mathcal{L})$ by the positive map $\mathcal{L}: C(\mathbb{A}) \rightarrow C(\mathbb{A})$ of (2.11). In particular, there exists $S \in C^*(\mathbb{A}, \Gamma)$ such that*

$$S^*i_A(a)S = i_A(\mathcal{L}(a)) \quad (2.14)$$

for all $a \in A$.

Proof. The element S exists by [Kwa17, Definition 3.1] and [Kwa17, Theorem 3.13]. \square

Remark 2.6.7. The element $S \in C^*(\mathbb{A}, \Gamma)$ satisfying (2.14) can be described concretely. Let $1 \in E_\Gamma$ be the function $1(x, y) = 1$. Define $S := i_{E_\Gamma}(1)$. Since (i_A, i_{E_Γ}) is a representation and $(1 \mid \phi(a)1)_A = \mathcal{L}(a)$ for all $a \in A$,

$$S^*i_A(a)S = i_E(1)^*i_A(a)i_E(1) = i_A((1 \mid \phi(a)1)_A) = i_A(\mathcal{L}(a)).$$

We now turn our attention to the case of invertible systems, and the relationship between the associated shift map σ and \mathcal{L} . As it turns out, \mathcal{L} is a transfer operator for σ . Classically, transfer operators have many important applications in topological dynamics (see [Rue78] for instance). In the C^* -algebraic setting, transfer operators were introduced by Exel in [Exe03] to study crossed-products by endomorphisms.

Definition 2.6.8 ([Exe03, Definition 2.1]). A *transfer operator* for a $*$ -endomorphism $\alpha: A \rightarrow A$ of a unital C^* -algebra A is a positive linear map $\mathcal{L}: A \rightarrow A$ such that $\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b)$, for all $a, b \in A$. We call the triple (A, α, \mathcal{L}) an *Exel system*.

Lemma 2.6.9. *Let (\mathbb{A}, Γ) be an invertible iterated function system with shift map $\sigma: \mathbb{A} \rightarrow \mathbb{A}$. Let $\sigma^*: C(\mathbb{A}) \rightarrow C(\mathbb{A})$ be dual to the shift map. Then $\mathcal{L}: C(\mathbb{A}) \rightarrow C(\mathbb{A})$ is a transfer operator for σ^* . In particular, $(C(\mathbb{A}), \sigma^*, \mathcal{L})$ is an Exel system.*

Proof. For each $a, b \in A$ we compute,

$$\mathcal{L}(\sigma^*(a)b)(x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} a(\sigma \circ \gamma(x))b(\gamma(x)) = (a\mathcal{L}(b))(x). \quad \square$$

Exel introduced transfer operators in order to define a notion of crossed products by non-invertible endomorphisms of C^* -algebras. These crossed products are now known as *Exel crossed products* and we direct the reader to [Exe03] for the definition. Given an Exel system (A, α, \mathcal{L}) we denote the associated Exel crossed product by $A \times_{\alpha, \mathcal{L}} \mathbb{N}$.

Brownlowe and Raeburn ([BR06, Proposition 3.10] and the comment thereafter) showed that for a unital C^* -algebra A with unital $*$ -endomorphism α , the Exel crossed

product $A \times_{\alpha, \mathcal{L}} \mathbb{N}$ can be realised as the Cuntz-Pimsner algebra of a C^* -correspondence $M_{\mathcal{L}}$. As it turns out, the correspondence $M_{\mathcal{L}}$ is isomorphic to the KSGNS correspondence $F_{\mathcal{L}}$ (see [Kwa17, Lemma 4.4]). Moreover, any Exel system can be realised as a crossed-product by a completely positive map [Kwa17, Theorem 4.7].

In [dCa09, Theorem 3.22], de Castro shows that for an invertible contractive iterated function system satisfying either the open-set condition or $|B_{\Gamma}| < \infty$, the Kajiwara-Watatani algebra $C^*(\mathbb{A}, \Gamma)$ coincides with the Exel crossed product $C(\mathbb{A}) \rtimes_{\sigma^*, \mathcal{L}} \mathbb{N}$. With Kwaśniewski's framework we can easily extend de Castro's result to the general invertible setting. Indeed, Proposition 2.6.6 and Lemma 2.6.9 imply the following.

Proposition 2.6.10. *Let (\mathbb{A}, Γ) be an invertible iterated function system with shift map $\sigma: \mathbb{A} \rightarrow \mathbb{A}$ and transfer operator \mathcal{L} given by (2.11). Then the Kajiwara-Watatani algebra $C^*(\mathbb{A}, \Gamma)$ is isomorphic to the Exel crossed product $C(\mathbb{A}) \rtimes_{\sigma^*, \mathcal{L}} \mathbb{N}$.*

Combining Proposition 2.6.10 with Proposition 2.5.29 we see that each Kajiwara-Watatani algebra always sits naturally as a subalgebra of an Exel crossed product.

Corollary 2.6.11. *Let (\mathbb{A}, Γ) be an iterated function system with inverse lifted system $(\tilde{\mathbb{A}}, \tilde{\Gamma})$ (see Definition 2.5.19). Let $\tilde{\sigma}: \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ denote the shift map on $\tilde{\mathbb{A}}$ and let $\tilde{\mathcal{L}}$ denote the positive map from (2.11) associated to $(\tilde{\mathbb{A}}, \tilde{\Gamma})$. Then there is an injective unital $*$ -homomorphism $\Phi: C^*(\mathbb{A}, \Gamma) \rightarrow C(\tilde{\mathbb{A}}) \rtimes_{\tilde{\sigma}^*, \tilde{\mathcal{L}}} \mathbb{N}$. Moreover, $\Phi(i_A(A)) \subseteq i_{\tilde{A}}(\tilde{A})$, and letting $S = i_{E_{\Gamma}}(1_{\text{Gr}(\Gamma)})$ and $\tilde{S} = i_{E_{\tilde{\Gamma}}}(1_{\text{Gr}(\tilde{\Gamma})})$ we have $\Phi(S) = \tilde{S}$.*

Proof. The only thing that does not follow immediately from Proposition 2.6.10 and Proposition 2.5.29 is that $\Phi(S) = \tilde{S}$. However, since $\mu_2(1_{\text{Gr}(\Gamma)}) = 1_{\text{Gr}(\tilde{\Gamma})}$, we have

$$\Phi(S) = \Phi \circ i_{E_{\Gamma}}(1_{\text{Gr}(\Gamma)}) = i_{E_{\tilde{\Gamma}}} \circ \mu_2(1_{\text{Gr}(\Gamma)}) = \tilde{S}. \quad \square$$

We finish this section with an observation about iterated function systems and Cuntz algebras. Recall from Proposition 2.1.4 that if $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ is an iterated function system admitting a code map $\pi: \Omega_N \rightarrow \mathbb{A}$, then there is an isomorphism $\Phi: \mathcal{O}_{X_{\Gamma}} \rightarrow \mathcal{O}_N$. Moreover, Equation (2.5) implies that Φ acts as a noncommutative code map, in the sense that, if $\alpha: C(\Omega_N) \rightarrow \mathcal{O}_N$ is inclusion of the diagonal, then $(\Phi \circ i_{C(\mathbb{A})})(a \circ \gamma_i) = \alpha(\pi^*(a) \circ \bar{\gamma}_i)$. Recall also from Equation (2.3) that $S_i^* \alpha(f) S_i = \alpha(f \circ \bar{\gamma}_i)$ for all $f \in C(\Omega_N)$. Consequently,

$$S_i^* (\Phi \circ i_{C(\mathbb{A})})(a) S_i = (\alpha \circ \pi^*)(a \circ \gamma_i) \quad (2.15)$$

for all $a \in C(\mathbb{A})$. In this sense, conjugation by S_i implements the map γ_i^* on $C(\mathbb{A})$. It also follows that for all $a \in C(\mathbb{A})$ we have,

$$(\Phi \circ i_{C(\mathbb{A})})(\mathcal{L}(a)) = \frac{1}{N} \sum_{i=1}^N (\alpha \circ \pi^*)(a \circ \gamma_i) = \sum_{i=1}^N \frac{1}{N} S_i^* (\Phi \circ i_{C(\mathbb{A})})(a) S_i.$$

So the completely positive map $x \mapsto \frac{1}{N} \sum_{i=1}^N S_i^* x S_i$ on $(\Phi \circ i_{C(\mathbb{A})})(C(\mathbb{A}))$ implements \mathcal{L} , and by Proposition (2.6.1) the faithfulness of this map is equivalent to Γ -invariance of \mathbb{A} . This suggests the following definition.

Definition 2.6.12. A *noncommutative iterated function system* is a unital C^* -subalgebra A of \mathcal{O}_N such that

- (1) $S_i^*AS_i \subseteq A$ for all $1 \leq i \leq N$; and
- (2) the map $\mathcal{L}: A \rightarrow A$ defined by $\mathcal{L}(a) = \frac{1}{N} \sum_{i=1}^N S_i^*aS_i$ is a unital completely positive map on A .

A noncommutative iterated function system is *injective* if $S_i^*AS_i = A$ for all $1 \leq i \leq N$.

If A satisfies the hypotheses of Definition 2.6.12 and is also a subalgebra of the diagonal \mathcal{D}_N of \mathcal{O}_N , then A corresponds to an iterated function system in the sense of Definition 1.0.1. Condition (1) corresponds to the fact that each $\gamma \in \Gamma$ maps \mathbb{A} into \mathbb{A} , while condition (2) is equivalent to Γ -invariance of \mathbb{A} . The injectivity condition is equivalent to requiring that each $\gamma \in \Gamma$ is injective.

We do not dwell on the concept of noncommutative iterated function systems since the only interesting examples that we are aware of is when A is commutative, and there is still much to be done in this case.

Pinzari-Watatani-Yonetani [PWY00, §4.3] considered a different notion of noncommutative contractive iterated function system, however Ionescu [Ion07, Corollary 3.5] showed that such systems are necessarily commutative.

2.7 | A frame for E_Γ

Let $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_{|\Gamma|}\})$ be an iterated function system. In this section we consider frames for the module E_Γ . Kasparov's Stabilisation Theorem implies that every countably generated Hilbert module over a unital C^* -algebra admits a frame (see Theorem A.1.7). We have already used the existence of frames a number of times (for example Proposition 2.5.17). However, having a concrete frame for a Hilbert module facilitates computations, just as a well-chosen orthonormal basis facilitates computation in a Hilbert space.

We are able to construct a frame when (\mathbb{A}, Γ) is branch isolated (see Definition 2.7.8). This includes all examples where B_Γ is finite. In a general setting, constructing a frame seems to be a more subtle issue. Frames for E_Γ were considered in the preprint [KW04, §4] for contractive iterated function systems satisfying both the open-set condition, and finiteness of B_Γ . Our construction is similar to that of [KW04], but does not require the open-set condition. Examples which are covered in our framework but were not previously covered include Example 2.2.12.

Our main simplifying observation is that Proposition 2.6.4 implies that E_Γ splits as a direct sum $E_\Gamma = p_2^*(A) \oplus E_\Gamma^0$ of right Hilbert A -modules, where

$$E_\Gamma^0 = \{\xi \in C(\text{Gr}(\Gamma)) \mid \sum_{\gamma \in \Gamma} \xi(\gamma(y), y) = 0 \text{ for all } y \in \mathbb{A}\}.$$

Since the function $e_0(x, y) = 1$ defines a frame for $p_2^*(A) \cong A$, to find a frame for E_Γ it suffices to construct a frame $(e_i)_{i=1}^\infty$ for E_Γ^0 .

Recall that $G_\Lambda = \bigcap_{\gamma \in \Lambda} \text{Gr}(\gamma)$ whenever $\Lambda \subseteq \Gamma$. Note that if $(x, y) \in G_\Gamma$, then the definition of E_Γ^0 implies that $\xi(x, y) = 0$ for all $\xi \in E_\Gamma^0$. We begin by restricting attention to iterated function systems of a particular form.

Definition 2.7.1. An iterated function system (\mathbb{A}, Γ) is *uniformly branched* if $B_\Gamma = \partial G_\Gamma$.

Example 2.7.2. The iterated function system (\mathbb{A}, Γ') from Example 2.2.3 is uniformly branched as $B_\Gamma = G_{\{\gamma_1, \gamma'_2\}}$. \triangle

Suppose for now that (\mathbb{A}, Γ) is a uniformly branched iterated function system. Inspired by [KW04], fix a strictly increasing countable approximate unit $(u_m)_{m=0}^\infty$ for $C_0(\mathbb{A} \setminus p_2(G_\Gamma))$, such that $u_0 = 0$. Identify each u_m with its extension by zero in $C(\mathbb{A})$. For each $m \geq 1$ let $v_m = (u_m - u_{m-1})^{1/2}$. Note that $u_m = \sum_{k=1}^m v_k^2$ and $|u_m(x)| \leq 1$ for all $m \in \mathbb{N}$ and $x \in \mathbb{A}$.

Lemma 2.7.3. *For each open set $U \subseteq \mathbb{A} \setminus p_2(G_\Gamma)$ and each $\varepsilon > 0$ there exists $M_U \in \mathbb{N}$ such that for all $y \in \mathbb{A} \setminus U$ we have $|u_m(y) - 1| < \varepsilon$ and $v_m(y) < \varepsilon$ for all $m > M_U$.*

Proof. Use the Tietze Extension Theorem to choose $a \in C_0(\mathbb{A} \setminus p_2(G_\Gamma))$ such that $a(y) = 1$ for all y in the closed set $\mathbb{A} \setminus U$. Fix $\varepsilon > 0$ and take $N \in \mathbb{N}$ such that $m \geq N/2$ implies $\|u_m a - a\| < \varepsilon$. It follows that for $m \geq N/2$ we have $|u_m(y) - 1| < \varepsilon$ for all $y \in \mathbb{A} \setminus U$. For $m \geq N$, we have $|u_m(y) - u_{m+1}(y)| < \varepsilon$ for all $y \in \mathbb{A} \setminus U$. In particular, there exists $M_U \geq N$ such that $m \geq M_U$ implies $v_m(y) < \varepsilon$ for all $y \in \mathbb{A} \setminus U$. \square

Let ω be a principal $|\Gamma|$ -th root of unity, let $m \geq 1$, let $1 \leq l \leq |\Gamma| - 1$, and let $i = (m - 1)(|\Gamma| - 1) + l$. Recall the branch index $b(x, y)$ from Definition 2.2.13. Inspired by the discrete Fourier transform, we consider the function e_i on $\text{Gr}(\Gamma)$ defined by

$$e_i(x, y) := \frac{v_m(y)}{b(x, y)} \sum_{\{k | \gamma_k(y) = x\}} \omega^{lk}. \quad (2.16)$$

Recall that if ω is a principal $|\Gamma|$ -th root of unity then

$$\sum_{k=0}^{|\Gamma|-1} \omega^{lk} = \begin{cases} |\Gamma| & \text{if } l = 0; \\ 0 & \text{if } l \neq 0. \end{cases}$$

Lemma 2.7.4. *Let (\mathbb{A}, Γ) be a uniformly branched iterated function system. For each $m \geq 1$ and $1 \leq l \leq |\Gamma| - 1$ the function $e_{(m-1)(|\Gamma|-1)+l}$ defined by (2.16) belongs to E_Γ^0 .*

Proof. Let $i = (m - 1)(|\Gamma| - 1) + l$. For continuity, note that continuity of v_m implies that $e_i(x, y) = 0$ for all $(x, y) \in G_\Gamma$, and that for every $(x_0, y_0) \in \partial G_\Gamma$ we have $e_i(x_n, y_n) \rightarrow 0$ for any sequence such that $(x_n, y_n) \rightarrow (x_0, y_0)$.

By assumption $B_\Gamma = \partial G_\Gamma$. Hence, the multi-valued map $(x, y) \mapsto \{k \mid \gamma_k(y) = x\}$ is locally constant on the open set $\text{Gr}(\Gamma) \setminus G_\Gamma$. In particular, $(x, y) \mapsto b(x, y)$ is also locally constant. It follows from continuity of v_m that $e_i(x, y)$ is continuous at (x, y) for all $(x, y) \in \text{Gr}(\Gamma) \setminus G_\Gamma$. Hence, e_i is continuous.

To see that e_i belongs to E_Γ^0 we compute,

$$\sum_{(x', y) \in \text{Gr}(\Gamma)} b(x', y) e_i(x', y) = v_m(y) \sum_{(x', y) \in \text{Gr}(\Gamma)} \sum_{\{k | \gamma_k(y) = x'\}} \omega^{lk} = \sum_{k=0}^{|\Gamma|-1} \omega^{lk} = 0. \quad \square$$

Our aim is to show that $(e_i)_{i=1}^\infty$ defines a frame for E_Γ^0 when (\mathbb{A}, Γ) is uniformly branched. However, we first require a topological lemma.

Lemma 2.7.5. *Let (\mathbb{A}, Γ) be an iterated function system. Let $p_2: \text{Gr}(\Gamma) \rightarrow \mathbb{A}$ be the projection onto the second factor. Then for each open set $W \subseteq \text{Gr}(\Gamma)$ containing G_Γ , there is an open subset U of \mathbb{A} containing $p_2(G_\Gamma)$ such that $p_2^{-1}(U) \subseteq W$.*

Proof. Let $U = \bigcap_{\gamma \in \Gamma} p_2(W \cap \text{Gr}(\gamma))$. Then U is open since $p_2|_{\text{Gr}(\gamma)}$ is an open map for each $\gamma \in \Gamma$. Since $G_\Gamma \subseteq \text{Gr}(\gamma)$ for all $\gamma \in \Gamma$, and $G_\Gamma \subseteq U$, it follows that $p_2(G_\Gamma) \subseteq U$. Now suppose that $(x, y) \in p_2^{-1}(U)$. Fix $\gamma' \in \Gamma$ such that $(x, y) \in \text{Gr}(\gamma')$. Since $p_2^{-1}(U) = \bigcap_{\gamma \in \Gamma} p_2^{-1} \circ p_2(W \cap \text{Gr}(\gamma))$, we have $(x, y) \in p_2^{-1} \circ p_2(W \cap \text{Gr}(\gamma))$ for all $\gamma \in \Gamma$. In particular, $(x, y) \in p_2^{-1} \circ p_2(W \cap \text{Gr}(\gamma'))$. Since $p_2|_{\text{Gr}(\gamma')}$ is injective, it follows that $(x, y) \in W \cap \text{Gr}(\gamma') \subseteq W$. \square

Proposition 2.7.6. *Suppose that (\mathbb{A}, Γ) is a uniformly branched iterated function system. Then the functions $(e_i)_{i=1}^\infty$ of (2.16) constitute a frame for E_Γ^0 .*

Proof. Fix $\xi \in E_\Gamma^0$ and $\varepsilon > 0$. Recall that $\Gamma y = \{\gamma(x) \mid \gamma \in \Gamma\}$. Since $\xi|_{G_\Gamma} = 0$, uniform continuity of ξ gives an open neighbourhood W of G_Γ such that $|\xi(x, y)| < \varepsilon$ for all $(x, y) \in W$. Let U be an open neighbourhood of $p_2(G_\Gamma)$ such that $p_2^{-1}(U) \subseteq W$, as constructed in Lemma 2.7.5. For any $y \in U$ we have $|\xi(x, y)| < \varepsilon$ for all $x \in \Gamma y$. Let M_U be as in Lemma 2.7.3, so that for all $y \in U^c$ we have $|\sum_{m=1}^{M_U} v_m(y)^2 - 1| < \varepsilon$, and $v_m(y) < \varepsilon$ for all $m \geq M_U$.

First suppose that $y \in U$ and $x \in \Gamma y$. Fix $i_0 \in \mathbb{N}$ and write $i_0 = (m_0 - 1)(|\Gamma| - 1) + l_0$ for the unique $m_0 \in \mathbb{N}$ and $0 \leq l_0 \leq |\Gamma| - 1$. Since $y \in U$ we have,

$$\left| \sum_{i=1}^{i_0} (e_i \cdot (e_i | \xi)_A)(x, y) - \xi(x, y) \right| < \left| \sum_{i=1}^{i_0} e_i \cdot (e_i | \xi)_A(x, y) \right| + \varepsilon$$

We estimate,

$$\begin{aligned} & \left| \sum_{i=1}^{i_0} e_i \cdot (e_i | \xi)_A(x, y) \right| \\ &= \left| \frac{1}{|\Gamma|} \sum_{m=1}^{m_0-1} \sum_{l=1}^{|\Gamma|-1} \sum_{x' \in \Gamma y} b(x', y) e_{(m-1)(|\Gamma|-1)+l}(x, y) \overline{e_{(m-1)(|\Gamma|-1)+l}(x', y)} \xi(x', y) \right. \\ & \quad \left. + \frac{1}{|\Gamma|} \sum_{l=1}^{l_0} \sum_{x' \in \Gamma y} b(x', y) e_{(m_0-1)(|\Gamma|-1)+l}(x, y) \overline{e_{(m_0-1)(|\Gamma|-1)+l}(x', y)} \xi(x', y) \right| \\ &\leq \left| \frac{1}{|\Gamma|} \sum_{m=1}^{m_0-1} \sum_{l=1}^{|\Gamma|-1} \sum_{x' \in \Gamma y} \frac{v_m(y)^2 \xi(x', y)}{b(x, y)} \sum_{\{k | \gamma_k(y)=x\}} \sum_{\{k' | \gamma_k(y)=x'\}} \omega^{l(k-k')} \right| \\ & \quad + \left| \frac{1}{|\Gamma|} \sum_{l=1}^{l_0} \sum_{x' \in \Gamma y} \frac{v_{m_0}(y)^2 \xi(x', y)}{b(x, y)} \sum_{\{k | \gamma_k(y)=x\}} \sum_{\{k' | \gamma_k(y)=x'\}} \omega^{l(k-k')} \right|. \end{aligned}$$

For the first term we have,

$$\begin{aligned} & \left| \frac{1}{|\Gamma|} \sum_{m=1}^{m_0-1} \sum_{l=1}^{|\Gamma|-1} \sum_{x' \in \Gamma y} \frac{v_m(y)^2 \xi(x', y)}{b(x, y)} \sum_{\{k | \gamma_k(y)=x\}} \sum_{\{k' | \gamma_k(y)=x'\}} \omega^{l(k-k')} \right| \\ &\leq \frac{1}{|\Gamma| b(x, y)} \left(\sum_{m=1}^{m_0-1} v_m(y)^2 \right) \sum_{l=1}^{|\Gamma|-1} \sum_{\{k | \gamma_k(y)=x\}} |\omega^{kl}| \sum_{x' \in \Gamma y} |\xi(x', y)| \sum_{\{k' | \gamma_k(y)=x'\}} |\omega^{-lk}| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{|\Gamma|b(x,y)} \sum_{l=1}^{|\Gamma|-1} b(x,y) \sum_{x' \in \Gamma y} b(x',y) |\xi(x',y)| \\
&< \frac{|\Gamma|-1}{|\Gamma|} \varepsilon \sum_{x' \in \Gamma y} b(x',y) \\
&= (|\Gamma|-1)\varepsilon.
\end{aligned}$$

Since $v_{m_0}(y) \leq 1$, for the second term we perform a similar calculation:

$$\begin{aligned}
&\left| \frac{1}{|\Gamma|} \sum_{l=1}^{l_0} \sum_{x' \in \Gamma y} \frac{v_{m_0}(y)^2 \xi(x',y)}{b(x,y)} \sum_{\{k|\gamma_k(y)=x\}} \sum_{\{k'|\gamma_{k'}(y)=x'\}} \omega^{l(k-k')} \right| \\
&\leq \frac{v_{m_0}(y)^2}{|\Gamma|b(x,y)} \sum_{l=1}^{l_0} \sum_{x' \in \Gamma y} \sum_{\{k|\gamma_k(y)=x\}} \sum_{\{k'|\gamma_{k'}(y)=x'\}} |\omega^{l(k-k')}| |\xi(x',y)| \\
&\leq \frac{v_{m_0}(y)^2}{|\Gamma|b(x,y)} \sum_{l=1}^{l_0} b(x,y) \sum_{x' \in \Gamma y} b(x',y) |\xi(x',y)| \\
&< \frac{l_0}{|\Gamma|} \varepsilon \sum_{x' \in \Gamma y} b(x',y) \\
&< |\Gamma| \varepsilon.
\end{aligned}$$

Consequently, for all $y \in U$ and $i_0 \in \mathbb{N}$ we have

$$\left| \sum_{i=1}^{i_0} e_i \cdot (e_i \mid \xi)_A(x,y) - \xi(x,y) \right| < \varepsilon + (|\Gamma|-1)\varepsilon + |\Gamma|\varepsilon = 2|\Gamma|\varepsilon,$$

for all $x \in \Gamma y$.

Now suppose that $y \in \mathbb{A} \setminus U$ and $x \in \Gamma y$. Suppose that $i_0 = M(|\Gamma|-1)$ for some $M \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{i=1}^{i_0} e_i \cdot (e_i \mid \xi)_A(x,y) &= \frac{1}{|\Gamma|} \sum_{i=1}^{i_0} \sum_{x' \in \Gamma y} b(x',y) e_i(x,y) \overline{e_i(x',y)} \xi(x',y) \\
&= \frac{1}{|\Gamma|} \sum_{m=1}^M \sum_{l=1}^{|\Gamma|-1} \sum_{x' \in \Gamma y} \frac{v_m(y)^2 \xi(x',y)}{b(x,y)} \sum_{\{k|x=\gamma_k(y)\}} \sum_{\{k'|x'=\gamma_{k'}(y)\}} \omega^{l(k-k')} \\
&= \frac{1}{|\Gamma|} \left(\sum_{m=1}^M v_m(y)^2 \right) \sum_{l=1}^{|\Gamma|-1} \sum_{x' \in \Gamma y} \frac{\xi(x',y)}{b(x,y)} \sum_{\{k|x=\gamma_k(y)\}} \sum_{\{k'|x'=\gamma_{k'}(y)\}} \omega^{l(k-k')}.
\end{aligned}$$

Using at the second equality that,

$$\sum_{l=0}^{|\Gamma|-1} \omega^{l(k-k')} = \begin{cases} 0 & \text{if } k \neq k' \\ |\Gamma| & \text{if } k = k', \end{cases}$$

we have

$$\begin{aligned}\xi(x, y) &= \frac{1}{|\Gamma|b(x, y)}\xi(x, y)|\Gamma|b(x, y) \\ &= \frac{1}{|\Gamma|b(x, y)} \sum_{x' \in \Gamma y} \xi(x', y) \sum_{\{k|x=\gamma_k(y)\}} \sum_{\{k'|x'=\gamma_{k'}(y)\}} \sum_{l=0}^{|\Gamma|-1} \omega^{l(k-k')}.\end{aligned}\tag{2.17}$$

Since $\xi \in E_\Gamma^0$, the $l = 0$ terms of (2.17) contribute nothing because

$$\sum_{x' \in \Gamma y} \xi(x', y) \sum_{\{k|x=\gamma_k(y)\}} \sum_{\{k'|x'=\gamma_{k'}(y)\}} 1 = b(x, y) \sum_{x' \in \Gamma y} b(x', y)\xi(x', y) = 0.$$

As such,

$$\sum_{i=1}^{i_0} e_i \cdot (e_i | \xi)_A(x, y) = \left(\sum_{m=1}^M v_m(y)^2 \right) \xi(x, y).$$

Recall from Lemma 2.7.3 that for all $y \in \mathbb{A} \setminus U$ we have $|\sum_{m=1}^M v_m(y)^2 - 1| < \varepsilon$ and $v_M(y) < \varepsilon$ for all $M \geq M_U$. It follows that if i_0 is of the form $i_0 = M(|\Gamma| + 1) + l_0$ for some $M \geq M_U$ and $1 \leq l_0 \leq |\Gamma| - 1$, then for $y \in \mathbb{A} \setminus U$ and $x \in \Gamma y$ we have

$$\begin{aligned}& \left| \sum_{i=1}^{i_0} e_i \cdot (e_i | \xi)_A(x, y) - \xi(x, y) \right| \\ & \leq \left| \sum_{i=1}^{M(1-|\Gamma|)} e_i \cdot (e_i | \xi)_A(x, y) - \xi(x, y) \right| \\ & \quad + \left| \frac{1}{|\Gamma|} \sum_{l=1}^{l_0} \sum_{x' \in \Gamma y} \frac{v_M(y)^2 \xi(x', y)}{b(x, y)} \sum_{\{k|\gamma_k(y)=x\}} \sum_{\{k'|\gamma_{k'}(y)=x'\}} \omega^{l(k-k')} \right| \\ & < \varepsilon + \frac{|v_M(y)|^2 \|\xi\|}{|\Gamma|b(x, y)} \sum_{l=1}^{l_0} \sum_{\{k|\gamma_k(y)=x\}} \sum_{x' \in \Gamma y} \sum_{\{k'|\gamma_{k'}(y)=x'\}} |\omega^{l(k-k')}| \\ & < \varepsilon + \frac{\varepsilon \|\xi\| l_0 |\Gamma| b(x, y)}{|\Gamma| b(x, y)} \\ & < \varepsilon + |\Gamma| \|\xi\| \varepsilon.\end{aligned}$$

It now follows that there exists $I \in \mathbb{N}$ large enough so that for all $i_0 \geq I$, we have $|\sum_{i=1}^{i_0} e_i \cdot (e_i | \xi)_A(x, y) - \xi(x, y)| < \varepsilon$ for all $(x, y) \in \text{Gr}(\Gamma)$. Since the module norm on E_Γ and the uniform norm on $\text{Gr}(\Gamma)$ are equivalent by Lemma 2.2.8, it follows that $(e_i)_{i=1}^\infty$ is a frame for E_Γ^0 . \square

Corollary 2.7.7. *Suppose that (\mathbb{A}, Γ) is a uniformly branched iterated function system. Then the functions $(e_i)_{i=0}^\infty$ define a frame for E_Γ , where $e_0(x, y) = 1$ for all $(x, y) \in \text{Gr}(\Gamma)$ and $(e_i)_{i=1}^\infty$ is the frame from Proposition 2.7.6.*

Proof. This follows from the direct sum decomposition $E_\Gamma = p_2^*(A) \oplus E_\Gamma^0$ and Proposition 2.7.6. \square

In a similar manner to [KW04, Theorem 4.3], the frame constructed in Corollary 2.7.7 can be easily extended to a more general class of iterated function systems.

Definition 2.7.8. An iterated function system (\mathbb{A}, Γ) is said to be *branch isolated* if there exist finite collections $\{U_i\}_{i=1}^I$ and $\{V_k\}_{k=1}^K$ of open sets such that, putting $W_{i,k} = (V_k \times U_i) \cap \text{Gr}(\Gamma)$, we have

- (i) $\overline{U_i} \cap \overline{U_j} = \emptyset$ for all $i \neq j$;
- (ii) $\overline{V_k} \cap \overline{V_l} = \emptyset$ for all $k \neq l$;
- (iii) for each $(x, y) \in B_\Gamma$ there exist i, k such that $(x, y) \in W_{i,k}$;
- (iv) for each $1 \leq k \leq K$ and $1 \leq i \leq I$ the intersection $W_{i,k} \cap B_\Gamma$ is a closed subset of ∂G_Λ for some $\Lambda \subseteq \Gamma$ with $|\Gamma| \geq 2$ (the intersection could be empty).

Remark 2.7.9. The branch isolated condition is somewhat contrived and could likely be generalised, but the need for such a generalisation is not clear. Note that if B_Γ is finite then (\mathbb{A}, Γ) is branch isolated.

Suppose that (\mathbb{A}, Γ) is branch isolated with open sets $\{U_i\}_{i=1}^I$ and $\{V_k\}_{k=1}^K$ and $W_{i,k}$ as in Definition 2.7.8. Then for $1 \leq i \leq I$ the set $\bigcup_{k=1}^K p_2(W_{i,k} \cap B_\Gamma)$ is a closed subset of U_i . Using normality, we let U_i^0 be an open subset of U_i containing $\bigcup_{k=1}^K p_2(W_{i,k} \cap B_\Gamma)$ such that $\overline{U_i^0} \subseteq U_i$. Let $U_{I+1} = \bigcup_{i=1}^I \mathbb{A} \setminus \overline{U_i^0}$, then $\{U_i\}_{i=1}^{I+1}$ is an open cover of \mathbb{A} .

For each $1 \leq i \leq I$ and $1 \leq k \leq K$ let $A_i = C(\overline{U_i})$ and $E_{i,k} = C(\overline{W_{i,k}})$. Then $E_{i,k}$ becomes a right A_i -module with the structure inherited from E_Γ . By construction, the set $X_i := p_2^{-1}(U_i) \setminus \bigcup_{k=1}^K \overline{W_{i,k}}$ is open and satisfies $\overline{X_i} \cap \overline{W_{i,k}} = \emptyset$ for all $1 \leq k \leq K$. In particular, $\overline{X_i} \cap B_\Gamma = \emptyset$. Let $E_{i,K+1} = C(\overline{X_i})$ with the right A_i -module structure it also inherits from E_Γ . It now follows that

$$E_i := \bigoplus_{k=1}^{K+1} E_{i,k}$$

is a right Hilbert A_i -module. By construction $p^{-1}(\overline{U_{I+1}})$ contains no branched points. We define $E_{I+1} = C(p^{-1}(\overline{U_{I+1}}))$, which we equip with a right A_{I+1} -module structure inherited from E_Γ .

By construction, each $E_{i,k}$ is isomorphic to a module coming from a uniformly branched iterated function system. In particular for each i, k there is a frame $(e_n^{i,k})_{n=1}^\infty$ of the form arising from Corollary 2.7.7. Consequently, $(e_n^{i,k})_{n,k}$ defines a frame for E_i . The module E_{I+1} admits a finite frame which we denote by $(e_n^{I+1,1})_n$.

Now let $\{\rho_i\}_{i=1}^{N+1}$ be a partition of unity subordinate to the open cover $\{U_i\}_{i=1}^I$. Let $u_n^{i,k}(x, y) = \rho_i(y)^{1/2} e_n^{i,k}(x, y)$ for $(x, y) \in \text{Gr}(\Gamma)$. The factor of $\rho_i(y)$ ensures that $u_n^{i,k}$ is a well-defined function in E_Γ . We now have the following result, with the proof following almost identically to the proof of [KW04, Theorem 4.3].

Theorem 2.7.10. *Let (\mathbb{A}, Γ) be a branch isolated iterated function system. Then the collection $(u_n^{i,k})_{i,k,n}$ defines a frame for E_Γ .*

Proof. Fix $\xi \in E_\Gamma$. Since $\text{supp}(\xi \cdot \rho_i) \subseteq p_2^{-1}(\overline{U_i})$, and $(e_n^{i,k})_{k,n}$ is a frame for E_i ,

$$\sum_{k=1}^{K+1} \sum_{n=1}^\infty e_n^{i,k} \cdot (e_n^{i,k} \mid \xi \cdot \rho_i)_{A_i} = \xi \cdot \rho_i.$$

Thus,

$$\sum_{i=1}^I \sum_{k=1}^{K+1} \sum_n u_n^{i,k} \cdot (u_n^{i,k} \mid \xi)_A = \sum_{i=1}^I \sum_{k=1}^{K+1} \sum_n e_n^{i,k} \cdot (e_n^{i,k} \mid \xi \cdot \rho_i) = \sum_{i=1}^I \xi \cdot \rho_i = \xi,$$

so $(u_n^{i,k})_{i,k,n}$ is a frame for E_Γ . □

Remark 2.7.11. Example 2.4.4 and Example 2.4.7 are both examples of iterated function systems which are not branch isolated, and so our frame construction does not apply. Example 2.4.7 in particular is fairly tame as it is contractive and satisfies the open-set condition. However, it is not clear to the author how to adapt either the arguments of Proposition 2.7.6 or the partition of unity argument to this relatively simple example. It seems that a completely new strategy is required for concretely constructing a frame for E_Γ (or E_Γ^0) in general.

CHAPTER 3

Topological Quivers and Associated Groupoids

This chapter originates from the author's investigation into the feasibility of constructing a natural groupoid from an iterated function system, seeking to recover the Kajiwara-Watatani algebra of Chapter 2. Naturality in this context is something of a philosophical concept, but such a groupoid should, morally, be constructed by using the dynamics of the iterated function system on the underlying attractor. For the most part this chapter is not focused on iterated function systems, but instead the relationship between Cuntz-Pimsner algebras and groupoid C^* -algebras.

Since Renault's thesis [Ren80], groupoids have become a mainstay in the toolkit of operator algebraists, particularly for those interested in dynamical systems. Many large classes of C^* -algebras associated to dynamical systems admit groupoid models. This includes Cuntz algebras [Ren80], directed graph C^* -algebras [Kum+97], and C^* -algebras associated to Smale spaces [Put96]. The advantage of having a groupoid model for a dynamical C^* -algebra is that its invariants (eg. K -theory, KMS-states) are often computable in terms of the underlying groupoid. Such invariants can often then be interpreted in terms of properties of the original dynamical system. We also mention Li's [Li20] recent work in which he showed that every classifiable simple C^* -algebra can be realised as a twisted groupoid C^* -algebra. Relevant background and notation for groupoids can be found in Appendix B or in Dana Williams' recent book [Wil19].

A source of inspiration for such a groupoid model is the class of topological graphs. Topological graphs and their C^* -algebras were introduced Katsura in a series of papers [Kat04a; Kat06a; Kat06b; Kat08]. Topological graph C^* -algebras were originally defined as Cuntz-Pimsner algebras, and they generalise the well-studied directed graph C^* -algebras [Kum+97; Rae05; EW80].

A topological graph $E = (E^0, E^1, r, s)$ consists of second-countable locally compact Hausdorff spaces E^0 and E^1 , together with a local homeomorphism $s: E^1 \rightarrow E^0$ and a continuous map $r: E^1 \rightarrow E^0$ (see Example 3.1.11). Much like Kajiwara-Watatani algebras, the C^* -algebra of a topological graph is the Cuntz-Pimsner algebra of a C^* -correspondence with commutative coefficient algebra.

Groupoids associated to the Toeplitz and Cuntz-Pimsner algebras of a topological graph were introduced by Katsura [Kat06a, §10.3] and expanded on by Yeend [Yee07] in the general setting of topological k -graphs. These groupoids are of Deaconu-Renault type and are associated to a singly generated dynamical system built from the topological graph. We discuss such groupoids in more detail in Section 3.1.4. In the particular case

where E is a row-finite topological graph with no sources, the associated *boundary path groupoid* is given by

$$\mathcal{G}_E = \{(x, n - m, y) \in E^\infty \times \mathbb{Z} \times E^\infty \mid \sigma^n(x) = \sigma^m(y)\}.$$

Here, E^∞ is the space of infinite paths in E —topologised as a subspace of $\prod_{i=1}^\infty E^1$ —and $\sigma: E^\infty \rightarrow E^\infty$ is the left-shift map. Yeend [Yee06] showed that the full groupoid algebra $C^*(\mathcal{G}_E)$ is isomorphic to the Cuntz-Pimsner algebra associated to E .

One of the key ingredients required to build a C^* -algebra from a groupoid is a choice of Haar system. For the boundary path groupoid, the requirement that $s: E^1 \rightarrow E^0$ is a local homeomorphism implies that the shift σ is a local homeomorphism. This in turn implies that \mathcal{G}_E is étale, and admits a Haar system consisting of counting measures.

As a naive attempt to replicate the construction of \mathcal{G}_E for an iterated function system (\mathbb{A}, Γ) , we could replace E^∞ with the attractor of the inverse lifted system

$$\tilde{\mathbb{A}} = \{(x_1, x_2, \dots) \in \mathbb{A}^\mathbb{N} \mid \forall i \in \mathbb{N}, x_i \in \Gamma x_{i+1}\}$$

of Section 2.5.2. The shift map $\tilde{\sigma}: \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ is still a continuous surjection, but Proposition 2.5.22 and Proposition 2.5.8 imply that due to the existence of branched points, $\tilde{\sigma}$ may not be a local homeomorphism. Therefore, it is no longer obvious if such a Haar groupoid can be constructed for (\mathbb{A}, Γ) . We cannot just excise the branched points since they contain vital information about the algebra $C^*(\mathbb{A}, \Gamma)$. This is not to say that a natural groupoid model for $C^*(\mathbb{A}, \Gamma)$ does not exist, but it is likely that such a groupoid will not be étale.

To determine precisely the issues faced in the iterated function system case we work in the extended framework of topological quivers. Topological quivers were introduced by Muhly and Solel in [MS00] and further expanded upon by Muhly and Tomforde in [MT05b]. Topological quivers generalise topological graphs: the condition that s is a local homeomorphism is replaced by a family of Radon measures on E^1 fibred over s . With this relaxation on s , topological quiver algebras provide a joint generalisation of both topological graph algebras and Kajiwara-Watatani algebras. Thus, topological quivers provide a natural setting for comparing and contrasting the two classes of C^* -algebras.

Our goal is then to find a systematic way to construct a groupoid model for topological quiver C^* -algebra. Such a groupoid should agree with the boundary path groupoid in the case of topological graphs. To simplify matters we restrict ourselves to the simpler problem of finding a groupoid model for the core topological quiver C^* -algebra. In Section 3.2, we outline a strategy for constructing such a groupoid. We note that with very restrictive hypotheses, the cores of Kajiwara-Watatani algebras were examined in [KW14; KW16].

Unfortunately, we have been unable to complete our programme for a general topological quiver algebra. We pinpoint exactly where the existence of branched points acts as an obstruction in Remark 3.3.18 and Remark 3.3.22. Our inability to construct a natural groupoid for a topological quiver is not entirely surprising. In the introduction of [MT05b] Muhly and Tomforde cite one of their motivations for introducing topological quivers was due to the difficulty in associating a Haar groupoid to the branched coverings of Deaconu and Muhly [DM01], without first removing the branched points.

As a secondary outcome of this chapter we give a new—bottom-up—construction of the Deaconu-Renault groupoid associated to a topological graph. We start with a collection of simple equivalence relations defined on a topological graph, and assemble them into

the Deaconu-Renault groupoid. Perhaps the most interesting part of this description are two new topological constructions which are used extensively in the process: *perfections* and *adjunction groupoids*. Both of these constructions are likely to find a use beyond the present scope, and will be subject to further research. Unfortunately, the topological arguments do get fairly technical at this point.

Perfections are a way to extend a continuous map to a perfect map^a in a way analogous to compactification. They are described in Appendix C.3. We use perfections to give a new construction of the boundary path space of a topological graph, which in some way explains its topology.

The adjunction groupoid construction allows two topological groupoids to be glued over a common subgroupoid. This construction is described in Appendix C.2. Adjunction groupoids are used to glue equivalence relations—built from paths in a topological graph—together in a compatible way.

The author also hopes that this chapter helps add to the discourse for going between the C^* -algebras of Deaconu-Renault-type groupoids, and Cuntz-Pimsner algebras, in way that is complementary to the results of [RRS17] and [DKM01].

3.1 | Topological quivers and their C^* -algebras

3.1.1 Topological quivers

Topological quivers were introduced by Muhly and Solel [MS00, Example 5.4] to build a general class of dynamical C^* -correspondences over commutative C^* -algebras. Topological quivers and their C^* -algebras were later studied in more depth by Muhly and Tomforde [MT05b], who explored much of the fundamental structure theory. Before we introduce topological quivers, we make the following definition.

Definition 3.1.1. Let X and Y be locally compact Hausdorff spaces and let $s: X \rightarrow Y$ be a continuous open map. An *s-system* is a family of positive Radon measures $\lambda = \{\lambda_y\}_{y \in Y}$ on X satisfying the following two conditions:

- (i) $\text{supp } \lambda_y = s^{-1}(y)$ for all $y \in Y$; and
- (ii) $y \mapsto \int_X \xi(x) d\lambda_y(x)$ belongs to $C_c(Y)$ for all $\xi \in C_c(X)$.

A Haar system—in the sense of Definition B.2.1—is a particular case of an *s-system*. We always assume that our *s-systems* consist of σ -finite measures.

Definition 3.1.2 ([MS00; MT05b]). A *topological quiver* $E = (E^0, E^1, r, s, \lambda)$ consists of second-countable locally compact Hausdorff spaces E^0 and E^1 —whose elements are called *vertices* and *edges*—together with a continuous map $r: E^1 \rightarrow E^0$ and a continuous open map $s: E^1 \rightarrow E^0$ —called the *range* and *source*—and an *s-system* $\lambda = \{\lambda_v\}_{v \in E^0}$.

Our definition swaps the role of the range and source maps found in [MT05b]. This choice aligns itself with the so-called “southern hemisphere” convention for graph C^* -algebras as found in [Rae05]. We will always assume that s is surjective to avoid issues relating to fullness of Hilbert modules. No such assumption is made about r .

^aA perfect map is a continuous, proper, surjection.

Dynamically speaking, it is perhaps best to think of a topological quiver as a type of multi-valued dynamical system on E^0 . Indeed, the quiver E can be thought of as mapping each $v \in E^0$ to the set of values $r \circ s^{-1}(v)$ in E^0 via the edges $x \in s^{-1}(v)$.

As in the case of directed graphs, a topological quiver does not need to be *multiplicity-free*: it is possible to have both $s(x) = s(y)$ and $r(x) = r(y)$ for $x \neq y$ in E^1 .

In the case where E is multiplicity-free, a topological quiver is just a topological relation with an s -system. The connection between multiplicity-free topological quivers and topological relations was studied by Brenken [Bre10b]. Topological quivers coincide with topological graphs when s is a local homeomorphism and the s -system is given by counting measure (see Example 3.1.11).

Dynamically speaking, for each $v \in E^0$ the measures λ_v may be thought of as assigning weights (though they may not be probability measures) to edges the $x \in s^{-1}(v)$ that v can flow along. The relation between topological quivers and Markov operators was studied by Ionescu-Muhly-Vega [IMV12]. Similar perspectives appear in the work of Dor-on [DO18] on weighted partial systems. In [Kwa17, §3.5] Kwaśniewski showed that many examples of topological quivers arise from, and give rise to, positive linear maps $\rho : C_0(E^0) \rightarrow C_0(E^0)$.

We record the following Lemma for future use.

Lemma 3.1.3. *Let X and Y be second-countable locally compact Hausdorff spaces. Let λ be an s -system for a continuous open map $s : X \rightarrow Y$. Then for each compact $K \subseteq X$, we have $\sup_{v \in Y} \lambda_v(K) < \infty$.*

Proof. It follows from Urysohn's Lemma [Rud87, Theorem 2.12] that we can find $f \in C_c(X)$ such that $f|_K = 1$. Since $v \mapsto \int_Y f d\lambda_v$ belongs to $C_c(Y)$, it follows that $\sup_{v \in Y} \lambda_v(f) < \infty$. Hence, $\sup_{v \in Y} \lambda_v(K) \leq \sup_{v \in Y} \lambda_v(f) < \infty$. \square

Given a topological quiver $E = (E^0, E^1, r, s, \lambda)$ we define, for each $n \in \mathbb{N}$ a new topological quiver $E^{(n)}$ from paths in E . We call

$$E^n := \{(x_1, \dots, x_n) \in E^1 \times \dots \times E^1 \mid s(x_i) = r(x_{i+1})\}$$

the collection of *paths of length n in E* . Equip E^n with the subspace topology inherited from $\prod^n E^1$. If $x \in E^n$ for some $n \in \mathbb{N}_0$ then we say that x has *length n* , denoted $\ell(x) = n$.

The range $r_n : E^n \rightarrow E^0$ is given by $r_n(x_1, \dots, x_n) = r(x_1)$ and the source $s_n : E^n \rightarrow E^0$ is given by $s_n(x_1, \dots, x_n) = s(x_n)$. The s_n -system $\lambda^n = \{\lambda_v^n\}_{v \in E^0}$ consists of measures λ_v^n on E^n defined inductively by

$$\lambda_v^n(f) = \int_{s_{n-1}^{-1}(v)} \int_{s^{-1}(r(x_2))} f(x_1, \dots, x_n) d\lambda_{r(x_2)}(x_1) d\lambda_v^{n-1}(x_2, \dots, x_n) \quad (3.1)$$

for all $f \in C_c(E^n)$. We usually write $x_1 \cdots x_n$ instead of (x_1, \dots, x_n) . By applying (3.1) inductively, it follows that

$$\lambda_v^n(f) = \int_{s_{n-1}^{-1}(v)} \int_{s^{-1}(r(x_{k+1}))} f(x_1 \cdots x_n) d\lambda_{r(x_{k+1})}(x_1 \cdots x_k) d\lambda_v^{n-k}(x_{k+1} \cdots x_n) \quad (3.2)$$

for all $k \leq n$.

Definition 3.1.4. We call $E^{(n)} := (E^0, E^n, r_n, s_n, \lambda^n)$ the *quiver of paths of length n associated to E* .

In [MT05b, p.25] it is shown that $E^{(n)}$ is indeed a topological quiver.

The space E^n can be considered as fibred product (see Section C.1) over shorter paths. Indeed, for all $n \geq 2$ and $k + l = n$ we have

$$E^n \simeq E^k \times_{s_k, r_l} E^l.$$

The subscripts on the range and source maps will be suppressed if the value of n is clear. If $x \in E^0$ then we define $r(x) = s(x) = x$.

Definition 3.1.5. We call the set

$$E^\infty := \left\{ x_1 x_2 \cdots \in \prod_{\mathbb{N}} E^1 \mid s(x_i) = r(x_{i+1}) \text{ for all } i \in \mathbb{N} \right\}$$

the collection of *infinite paths of E* .

Notation 3.1.6. Suppose that either $x \in E^0$, $x = x_1 \cdots x_n \in E^n$ for some $n \in \mathbb{N}$, or $x = x_1 x_2 \cdots \in E^\infty$. For each $0 \leq l \leq k \leq \infty$ we define

$$x_{[l,k]} = \begin{cases} x & \text{if } n = 0; \\ x_l x_{l+1} \cdots x_k & \text{if } k \leq n < \infty; \\ x_l x_{l+1} \cdots x_n & \text{if } l \leq n \leq k; \\ s(x) & \text{if } n < l; \\ x_l x_{l+1} \cdots & \text{if } l < k = n = \infty. \end{cases}$$

We leave the case where $l = k = n = \infty$ undefined since we have no sensible use for it.

3.1.2 The quiver correspondence

Let $E = (E^0, E^1, r, s, \lambda)$ be a topological quiver. Following [MT05b], we associate a C^* -correspondence X_E to E . Consider the collection $C_c(E^1)$ of compactly supported, continuous functions on E^1 and let $A := C_0(E^0)$. Define a right action of $a \in A$ on $\xi \in C_c(E^1)$ by

$$(\xi \cdot a)(x) = \xi(x)a(s(x)).$$

Conditions (i) and (ii) of Definition 3.1.2 imply that for $\xi, \eta \in C_c(E^1)$ the formula

$$(\xi \mid \eta)_A(v) := \int_{E^1} \overline{\xi(x)} \eta(x) d\lambda_v(x)$$

defines a right A -valued inner product on $C_c(E^1)$, turning $C_c(E^1)$ into a right inner product A -module. Let X_E be the right Hilbert A -module given by completing $C_c(E^1)$ in the norm induced by the inner product (see [RW98, Lemma 2.16]). Explicitly, the norm on X_E is given by

$$\|\xi\|^2 = \sup_{v \in E^0} \int_{E^1} |\xi(x)|^2 d\lambda_v(x).$$

Recall that a sequence $(\xi_n)_{n \in \mathbb{N}}$ in $C_c(E^1)$ converges in the inductive limit topology to a function $\xi \in C_c(E^1)$ if and only if $\xi_n \rightarrow \xi$ uniformly, and there exists a compact set

K such that $\text{supp}(\xi_n) \subseteq K$ for all $n \in \mathbb{N}$ (see [Wil19, Lemma C.1]). The module norm topology on $C_c(E^1)$ is coarser than the inductive limit topology.

Lemma 3.1.7. *Suppose that $\xi_n \rightarrow \xi$ is a convergent sequence in $C_c(E^1)$ with respect to the inductive limit topology. Then $\|\xi_n - \xi\| \rightarrow 0$.*

Proof. Take a compact $K \subseteq E^1$ such that $\text{supp}(\xi_n) \subseteq K$ for all $n \in \mathbb{N}$. Then Lemma 3.1.3 implies that

$$\|\xi_n - \xi\|^2 = \sup_{v \in E^0} \int_K |\xi_n(x) - \xi(x)|^2 d\lambda_v(x) \leq \|\xi_n - \xi\|_\infty^2 \sup_{v \in E^0} \lambda_v(K). \quad \square$$

We now define a left action of A on X_E . Let $a \in A$ and for each $\xi \in C_c(E^1)$ define $\phi(a)\xi \in C_c(E^1)$ by,

$$\phi(a)\xi(x) = a(r(x))\xi(x)$$

Then $\xi \mapsto \phi(a)\xi$ is norm bounded, so the formula for $\phi(a)$ extends to X_E . Moreover, $\phi(a)$ is adjointable for each $a \in A$ with $\phi(a)^* = \phi(a^*)$. Since $\|\phi(a)\| \leq \|a\|$, ϕ defines a $*$ -homomorphism $\phi: A \rightarrow \text{End}_A(X_E)$.

Paths in topological quivers and tensor powers of X_E are related in the following way.

Lemma 3.1.8 ([MT05b, Lemma 6.2]). *Let $E = (E^0, E^1, r, s, \lambda)$ be a topological quiver. For each $k, l \in \mathbb{N}$, consider the linear map $\Gamma: C_c(E^k) \odot_{C_c(E^0)} C_c(E^l) \rightarrow C_c(E^{l+k})$ defined by,*

$$\Gamma(\xi \odot \eta)(x_1 \cdots x_{l+k}) = \xi(x_1 \cdots x_l)\eta(x_{l+1} \cdots x_{l+k}),$$

for all $\xi \in C_c(E^k)$ and $\eta \in C_c(E^l)$. Then Γ extends to an isomorphism between the A - A -correspondences $X_{E^{(l)}} \otimes_A X_{E^{(k)}}$ and $X_{E^{(l+k)}}$. Moreover, $X_E^{\otimes n}$ and $X_{E^{(n)}}$ are isomorphic as A - A -correspondences.

Proof. The left and right actions are clearly preserved by Γ on the level of compactly supported continuous functions. Fix $\xi \in C_c(E^l)$ and $\eta \in C_c(E^k)$. Since $E^l \times_{s_l, r_k} E^k \simeq E^{l+k}$ it follows from Lemma C.1.1 that $\Gamma(\xi \odot \eta)$ is compactly supported and continuous. Let $\lambda^l(\xi) \in C_c(E^0)$ denote the function $w \mapsto \int_{E^l} \xi d\lambda_w^l$ and recall that ϕ is the left action of $C_c(E^0)$ on $C_c(E^k)$. Using (3.2) we see that for all $v \in E^0$,

$$\begin{aligned} \lambda_v^{l+k}(\Gamma(\xi \odot \eta)) &= \int_{E^{l+k}} \xi(x_1 \cdots x_l)\eta(x_{l+1} \cdots x_k) d\lambda_v^{l+k}(x_1 \cdots x_{l+k}) \\ &= \int_{E^k} \left(\int_{E^l} \xi(x_1 \cdots x_l) d\lambda_{r(x_{l+1})}^l \right) \eta(x_{l+1} \cdots x_k) d\lambda_v^k(x_{l+1} \cdots x_{l+k}) \\ &= \lambda_v^k(\phi(\lambda^l(\xi))\eta). \end{aligned}$$

It follows that Γ preserves inner products and is therefore isometric.

The Stone-Weierstrass Theorem implies that $\{\Gamma(\xi \odot \eta) \mid \xi \in C_c(E^l), \eta \in C_c(E^k)\}$ spans a dense subspace of $C_c(E^{l+k})$ in the inductive limit topology. Lemma 3.1.7 now implies that such functions are dense in X_E^{l+k} , so Γ extends to the desired isomorphism.

The final statement follows from a repeated application of the first. \square

3.1.3 The C^* -algebras of a topological quiver

Following [MT05b] we apply Katsura's [Kat04b] methodology to construct the Toeplitz algebra \mathcal{T}_{X_E} and Cuntz-Pimsner algebra \mathcal{O}_{X_E} of X_E (see Section A.3). Let

$(j_A, j_{X_E}): (\varphi, X_E) \rightarrow \mathcal{T}_{X_E}$ and $(i_A, i_{X_E}): (\varphi, X_E) \rightarrow \mathcal{O}_{X_E}$ denote the universal representations of (φ, X_E) .

Recall that the Toeplitz algebra \mathcal{T}_{X_E} can be represented as creation and annihilation operators on the Fock module $\mathcal{F}(X_E) = \bigoplus_{n=0}^{\infty} X_E^{\otimes n}$. Since Lemma 3.1.8 gives an isomorphism $X_E^{\otimes n} \cong X_{E^{(n)}}$, creation operators on $\mathcal{F}(X_E)$ can be thought of as extending function on paths of length n to functions on paths of length $n+1$. As the Toeplitz algebra is generated by creation operators on $\mathcal{F}(X_E)$ together with their corresponding annihilation operators; the dynamics of the topological quiver E is encoded by \mathcal{T}_{X_E} .

Recall that the Katsura ideal of X_E is given by $I_{X_E} := \phi^{-1}(\text{End}_A^0(X_E)) \cap \ker(\phi)^\perp$. The Cuntz-Pimsner algebra \mathcal{O}_{X_E} is the quotient of \mathcal{T}_{X_E} by the ideal generated by $\{i_{X_E}^{(1)}(\phi(a)) - i_A(a) \mid a \in I_{X_E}\}$. Following [MT05b], we describe I_{X_E} using special sets of vertices.

Definition 3.1.9. Let $(E^0, E^1, r, s, \lambda)$ be a topological quiver. We define the following sets of vertices:

- *sources*: $E_{src}^0 := E^0 \setminus \overline{r(E^1)}$;
- *finite receivers*: $E_{fr}^0 := \{v \in E^0 \mid \text{there exists a precompact open neighbourhood } V \text{ of } v \text{ such that } r^{-1}(\overline{V}) \text{ is compact}\}$;
- *infinite receivers*: $E_{inf}^0 := E^0 \setminus E_{fr}^0$;
- *étale edges*: $E_{ét}^1 := \{x \in E^1 \mid \text{there exists an open neighbourhood } U \text{ of } x \text{ such that } s|_U \text{ is injective}\}$;
- *branched edges*: $E_{br}^1 := E^1 \setminus E_{ét}^1$;
- *étale vertices*: $E_{ét}^0 := \{v \in E^0 \mid \text{there exists an open neighbourhood } U \text{ of } v \text{ with } r^{-1}(U) \subseteq E_{ét}^1\}$;
- *branched vertices*: $E_{br}^0 := E^0 \setminus E_{ét}^0$;
- *regular vertices*: $E_{reg}^0 := (E_{fr}^0 \cap E_{ét}^0) \setminus \overline{E_{src}^0} = E_{fr}^0 \cap E_{ét}^0 \cap \text{int}(\overline{r(E^1)})$;
- *singular vertices*: $E_{sing}^0 := E^0 \setminus E_{reg}^0 = E_{inf}^0 \cup E_{br}^0 \cup \overline{E_{src}^0}$;
- *r-regular vertices*: $E_{rreg}^0 := E_{fr}^0 \setminus \overline{E_{src}^0} = E_{fr}^0 \cap \text{int}(\overline{r(E^1)})$;
- *r-singular vertices*: $E_{rsing}^0 := E^0 \setminus E_{rreg}^0 = E_{inf}^0 \cup \overline{E_{src}^0}$.

If $*$ denotes any of the above subscripts (aside from br and $ét$), then we write E_*^n to mean paths in E^n whose source lies in E_*^0 .

The collections of vertices given in Definition 3.1.9 are a refinement of those found in [MT05b, Definition 3.14]. In [MT05b], both the branched vertices and infinite receivers were called infinite receivers. We distinguish branched vertices and infinite receivers as they play markedly different roles in the structure of the associated correspondences. The distinction is made clear in the sequel.

We have also adopted the terminology r -regular and r -singular to distinguish the sources and infinite receivers—which only depend on the range map—from the branched vertices—which depend on the interplay between the range and source maps. With these distinguished classes of vertices in hand, we describe the covariance ideal.

Proposition 3.1.10 ([MT05b, Proposition 3.15]). *Let E be a topological quiver with associated C^* -correspondence X_E and left action $\phi : A \rightarrow \text{End}_A(X_E)$. Then*

$$(i) \ker(\phi) = C_0(E_{src}^0); \text{ and}$$

$$(ii) \phi^{-1}(\text{End}_A^0(X_E)) = C_0(E_{fr}^0 \cap E_{\text{et}}^0).$$

In particular, the covariance ideal $I_{X_E} := \phi^{-1}(\text{End}_A^0(X_E)) \cap \ker(\phi)^\perp$ of X_E is given by $I_X = C_0(E_{reg}^0)$.

Before proceeding, we introduce some important classes of topological quivers.

Example 3.1.11 (Topological Graphs). Recall that a *topological graph* (E^0, E^1, r, s) as defined by Katsura [Kat04a] consists of second-countable locally compact Hausdorff spaces E^0 and E^1 , together with a continuous map $r : E^1 \rightarrow E^0$, and a local homeomorphism $s : E^1 \rightarrow E^0$. Recall that s being a local homeomorphism means that for each $x \in E^1$ there exists a neighbourhood U of x such that $s|_U$ is a homeomorphism onto $s(U)$. If E^0 and E^1 are both discrete spaces, then (E^0, E^1, r, s) is called a *directed graph*.

Since $s : E^1 \rightarrow E^0$ is a local homeomorphism, for each $v \in E^0$ we can take λ_v to be counting measure on the fibre $s^{-1}(v)$ (this essentially follows from [Kat04a, Lemma 1.4]). Then $E = (E^0, E^1, r, s, \lambda)$ is a topological quiver. The C^* -correspondence X_E is isomorphic to the correspondence $C_d(E^1)$ defined by in [Kat04a] (see [MT05b, Remark 3.4]), which itself generalises the correspondences associated to directed graphs [FMR03, Example 1.5]. In particular, \mathcal{O}_{X_E} is isomorphic to Katsura's [Kat04a] C^* -algebra associated to a topological graph.

Since $s : E^1 \rightarrow E^0$ is a local homeomorphism, for a topological graph it is always the case that $E_{br}^1 = \emptyset$. Accordingly, $E_{sing}^0 = E_{rsing}^0$ and $E_{reg}^0 = E_{rreg}^0$. The covariance ideal is given by $I_X = C_0(E_{rsing}^0)$.

In the directed graph C^* -algebra literature it is common to call a graph for which every vertex is a finite receiver *row-finite*, since the adjacency matrix of the graph has finitely many entries in each row. We carry the same definition over to topological graphs. \blacktriangle

Example 3.1.12 (Iterated function systems). Recall from Definition 1.0.1 that an iterated function system (\mathbb{A}, Γ) consists of a second-countable compact Hausdorff space \mathbb{A} , together with a finite collection of continuous maps Γ on \mathbb{A} such that

$$\mathbb{A} = \bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A}).$$

Let $E^0 = \mathbb{A}$ and $E^1 = \text{Gr}(\Gamma)$. Define $r : E^1 \rightarrow E^0$ and $s : E^1 \rightarrow E^0$ to be the projections $r(x, y) = x$ and $s(x, y) = y$. Then s is open by Lemma 2.2.5.

Recall from Definition 2.2.13 that for $(x, y) \in \text{Gr}(\Gamma)$, the branch index is given by

$$b(x, y) = \#\{\gamma \in \Gamma \mid x = \gamma(y)\}.$$

The branch index determines an s -system $\lambda = \{\lambda_y\}_{y \in E^0}$ on E^1 : for each $y \in \mathbb{A}$ define a measure λ_y on $\text{Gr}(\Gamma)$ by,

$$\lambda_y(f) = \frac{1}{|\Gamma|} \sum_{(x, y) \in \text{Gr}(\Gamma)} b(x, y) f(x, y) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma(y), y),$$

for $f \in C(\text{Gr}(\Gamma))$. Clearly (i) of Definition 3.1.1 is satisfied by λ , and (ii) follows from the continuity of each $\gamma \in \Gamma$. Consequently, $X_E = (\mathbb{A}, \text{Gr}(\Gamma), r, s, \lambda)$ is a topological quiver.

The correspondence X_E is the Kajiwara-Watatani correspondence E_Γ defined in Chapter 2, which was first considered by Kajiwara and Watatani in [KW06]. In particular, the topological quiver algebra \mathcal{O}_{X_E} is the Kajiwara-Watatani algebra $C^*(\mathbb{A}, \Gamma)$ of Section 2.2.

Since $\mathbb{A} = \bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A})$, it follows that $E_{src}^0 = \emptyset$. As $\text{Gr}(\Gamma)$ is compact, r is proper, so $E_{inf}^0 = \emptyset$. Thus, E has no r -singular vertices. On the other hand Proposition 2.2.19 implies that E_{br}^1 is the set of branched points B_Γ , so $I_X = C_0(E_{\acute{e}t}^0) = C_0(E^0 \setminus r(B_\Gamma))$. \triangleleft

Remark 3.1.13. The quiver associated to an iterated function system is in direct contrast to the quiver associated to a topological graph. A topological graph has r -singular vertices but no branched vertices, while the quiver associated to the Kajiwara-Watatani correspondence has branched vertices but no r -singular vertices.

We mention a special class of topological graphs associated to branched coverings.

Example 3.1.14 (Branched coverings). Following Deaconu and Muhly [DM01, Definition 2.1], let T be a second-countable locally compact Hausdorff space, and suppose that U is a dense open set in T . A continuous surjective map $\sigma: T \rightarrow T$ is said to be a *branched covering* if:

- (i) $\{\text{connected components of } \sigma^{-1}(V) \mid V \text{ open in } T\}$ is a basis for the topology if T ;
- (ii) $\sigma(U)$ is a dense open subset of T ;
- (iii) $\sigma(T \setminus U) = T \setminus \sigma(U)$; and
- (iv) $\sigma|_U$ is a local homeomorphism.

Since $\sigma|_U$ is a local homeomorphism, following [Kat06a, §10.2] we associate a topological graph $E = (E^0, E^1, r, s)$ to a branched covering $\sigma: T \rightarrow T$. Let $E^0 = T$ and $E^1 = U$, $s = \sigma$, and let r be the natural inclusion. Then [Kat04a, Proposition 3.9] implies that \mathcal{O}_{X_E} is isomorphic to the augmented Cuntz-Pismner algebra of [DM01, Theorem 3.2].

Invertible iterated function systems (see Definition 2.5.1) give rise to branch coverings. If (\mathbb{A}, Γ) is an invertible iterated function system and $\sigma: \mathbb{A} \rightarrow \mathbb{A}$ is the corresponding shift map, then taking $T = \mathbb{A}$ and $U = \mathbb{A} \setminus p_1(B_\Gamma)$ yields a branched covering. The C^* -algebra associated to this branched covering differs from the Kajiwara-Watatani algebra of (\mathbb{A}, Γ) . If (\mathbb{A}, Γ') is the tent map system of Example 2.2.3, then $K_0(C^*(\mathbb{A}, \Gamma)) \cong \mathbb{Z}$ by Proposition 2.4.2. On the other hand if E is the topological graph arising from the branched covering associated to (\mathbb{A}, Γ) , then $K_0(\mathcal{O}_{X_E}) \cong \mathbb{Z}^2$ as computed in [DM01, Example 4.2]. \triangleleft

A topological quiver induces a continuous field of Hilbert spaces over E^0 . As a right A -module X_E can be considered as a completion of the continuous sections of this field. Moreover, the compact operators on X_E can be identified with the section algebra of an associated continuous field of elementary C^* -algebras. We direct the reader to [Dix77] for a comprehensive account of continuous fields of Hilbert spaces and continuous fields of elementary C^* -algebras, including their definitions.

Lemma 3.1.15 ([MT05b, Remark 3.2]). *Let $E = (E^0, E^1, r, s, \lambda)$ be a topological quiver. For each $v \in E^0$ let $\mathcal{H}_v := L^2(s^{-1}(v), \lambda_v)$. Then there is a continuous field of Hilbert spaces $\mathcal{H} := (\{\mathcal{H}_v\}_{v \in E^0}, \Gamma)$ such that X_E is isomorphic as a right Hilbert A -module to*

$$C_0(E^0, \mathcal{H}) := \{\xi \in \Gamma \mid v \mapsto \|\xi(v)\| \text{ is in } C_0(E^0)\}.$$

Moreover, if $\mathcal{K}_v := \mathcal{K}(L^2(\mathcal{H}_v))$ and $\mathcal{K} = (\{\mathcal{K}_v\}_{v \in E^0}, \Theta)$ is the continuous field of elementary C^* -algebras associated to \mathcal{H} , then $\text{End}_A^0(X_E)$ is isomorphic as a C^* -algebra to the section algebra,

$$C_0(E^1, \mathcal{K}) := \{T \in \Theta \mid v \mapsto \|T(v)\| \text{ is in } C_0(E^0)\}.$$

Remark 3.1.16. Although we do not present a proof of Lemma 3.1.15, we do describe the spaces Γ and Θ of continuous sections. For each $\xi \in C_c(E^1)$ let $\xi|_{s^{-1}v}$ denote the restriction of ξ to \mathcal{H}_v , and define $f_\xi: E^0 \rightarrow \bigsqcup_{v \in E^0} \mathcal{H}_v$ by $f_\xi(v) = \xi|_{s^{-1}(v)}$. Define a family of sections $\Lambda := \{f_\xi \mid \xi \in C_c(E^1)\} \subseteq \prod_{v \in E^0} \mathcal{H}_v$. By [Dix77, Proposition 10.2.3] there is a unique subset Γ of $\prod_{v \in E^0} \mathcal{H}_v$ containing Λ such that $\mathcal{H} := (\{\mathcal{H}_v\}_{v \in E^0}, \Gamma)$ defines a continuous field of Hilbert spaces. Define $\Theta_{f_\xi, f_\eta}: E^0 \rightarrow \bigsqcup_{v \in E^0} \mathcal{K}(\mathcal{H}_v)$ by $\Theta_{f_\xi, f_\eta}(v) = \Theta_{f_\xi(v), f_\eta(v)}$. The space Θ is obtained by applying [Dix77, Proposition 10.2.3] to $\text{span}\{\Theta_{f_\xi, f_\eta} \mid \xi, \eta \in C_c(E^1)\}$.

Under the identification of $\text{End}_A^0(X_E)$ with $C_0(E^1, \mathcal{K})$, an operator $T \in \text{End}_A^0(X_E)$ is identified with the unique element $T' \in C_0(E^1, \mathcal{K})$ that satisfies $(T'(v)f_\xi(v) \mid f_\eta(v))_{\mathbb{C}} = (T\xi \mid \eta)_A(v)$ for all $v \in E^0$ and $\xi, \eta \in C_c(E^1)$. We will only use this description of the compact operators on X_E in the proof of Proposition 3.2.14.

We finish section by showing how ideals of $C_0(E^0)$ give rise to submodules of X_E . Suppose that U is an open subset of E^0 , so that $J = C_0(U)$ is an ideal in $C_0(E^0)$. Since $s^{-1}(U)$ is open in E^1 we can view $C_c(s^{-1}(U))$ as a subspace of $C_c(E^1)$. Consider $C_c(s^{-1}(U))$ as a right pre-Hilbert J -module with right action, and right J -valued inner product induced from the one on $C_c(E^1)$. The closure of $C_c(s^{-1}(U)) \subseteq C_0(E^1)$ in the induced norm, is a right Hilbert J -module X_U .

Lemma 3.1.17. *Let $E = (E^0, E^1, r, s, \lambda)$ be a topological quiver. Suppose that U is an open subset of E^0 and $J = C_0(U)$. Then the right Hilbert J -modules X_U and $X_E \cdot J$ are isomorphic.*

Proof. By [Kat07, Proposition 1.3], $X_E \cdot J = \{\xi \in X_E \mid (\xi \mid \xi)_{X_E} \in J\}$. For each $\xi \in C_c(s^{-1}(U))$ we have $(\xi \mid \xi)_{X_U} \in J$. Since $C_c(s^{-1}(U))$ is a subspace of $C_c(E^1)$ and the inner products and right actions for X_U and X_E agree on $C_c(s^{-1}(U))$, it follows that X_U isometrically embeds in X_E .

Suppose that $\xi \in C_c(E^1)$ satisfies $(\xi \mid \xi)_{X_E} \in J$. Condition (ii) of Definition 3.1.1 implies that $(\xi \mid \xi)_{X_E} \in C_c(E^0)$. Consequently, $(\xi \mid \xi)_{X_E} \in C_c(U)$. Then for all $v \in E^0 \setminus \text{supp}((\xi \mid \xi)_{X_E})$,

$$\int_{s^{-1}(v)} |\xi(x)|^2 d\lambda_v(x) = 0.$$

Since $|\xi|^2$ is compactly supported, and λ_v is a Radon measure with full support, it follows that $\xi(x) = 0$ for all $x \in s^{-1}(v)$. It now follows that $\xi \in \overline{C_c(s^{-1}(U))} = X_U$. \square

3.1.4 Groupoids and topological graphs

There is a close relation between topological graphs and groupoids. We briefly describe the correspondence in this section. A short introduction to groupoids and their C^* -algebras can be found in Appendix B.

Definition 3.1.18 ([Ren00, Definition 2.4]). A *partial local homeomorphism* on a locally compact Hausdorff space T is a local homeomorphism $\sigma: \text{dom}(\sigma) \rightarrow \text{ran}(\sigma)$, where

$\text{dom}(\sigma)$ and $\text{ran}(\sigma)$ are open subsets of T . Such a pair (T, σ) is called a *singly generated dynamical system*. We associate to (T, σ) the *Deaconu-Renault groupoid*,

$$G(T, \sigma) := \{(x, m - n, y) \mid m, n \in \mathbb{N}_0, x \in \text{dom}(\sigma^m), y \in \text{dom}(\sigma^n), \sigma^m(x) = \sigma^n(y)\}.$$

Multiplication in the Deaconu-Renault groupoid is given by $(x, m - n, y)(y, n - k, z) = (x, m - k, z)$, which is only defined on pairs of this form. Inversion is given by $(x, m - n, y)^{-1} = (y, n - m, x)$. The unit space of $G(T, \sigma)$ can be identified with T , and under this identification $r(x, m - n, y) = x$ and $s(x, m - n, y) = y$.

A topology on $G(T, \sigma)$ is generated by basic open sets of the form

$$\mathcal{Z}(U, m, n, V) = \{(x, m - n, y) \in G(X, \sigma) \mid x \in U, y \in V, \sigma^m(x) = \sigma^n(y)\}, \quad (3.3)$$

ranging over all $n, m \in \mathbb{N}_0$ and $U \subseteq \text{dom}(\sigma^m)$, $V \subseteq \text{dom}(\sigma^n)$ open. Equipped with this topology $G(T, \sigma)$ becomes a locally compact Hausdorff groupoid which is both étale and amenable [Ren00, Propostion 2.4].

Given a singly generated dynamical system (T, σ) one can form a topological graph. Following [Kat06a] we take $E^0 = T$, $E^1 = \text{dom}(\sigma)$, $s = \sigma$, and let r the natural embedding of $\text{dom}(\sigma)$ into T . Then (E^0, E^1, r, s) is a topological graph, and the Cuntz-Pimsner algebra \mathcal{O}_{X_E} is isomorphic to $C^*(T, \sigma)$ by [Kat06a, Proposition 10.9].

Conversely, the Cuntz-Pimsner algebra of a topological graph can be modelled using a Deaconu-Renault groupoid of certain singly generated dynamical systems. We first introduce the unit space.

Definition 3.1.19. The *path space* of a topological graph E is the set

$$E^{\leq \infty} := E^\infty \sqcup \bigsqcup_{k=0}^{\infty} E^k,$$

and the *boundary path space* of E is the set

$$\partial E := E^\infty \sqcup \bigsqcup_{k=0}^{\infty} \{x \in E^k \mid r(x) \in E_{\text{sing}}^0\}.$$

Both $E^{\leq \infty}$ and ∂E can be equipped with a locally compact Hausdorff topology, but delay the description of this topology until Section 3.3.1 where it is discussed in more detail. The definition of the boundary path space we give differs from Yeend [Yee07]. However, it is shown in [KL17, Proposition 4.6] that the two definitions are equivalent. We now introduce the partial local homeomorphism from which we construct a singly generated dynamical system.

Lemma 3.1.20 ([KL17, Proposition 7.1]). *Let E be a topological graph. Define the left-shift $\sigma: E^{\leq \infty} \rightarrow E^{\leq \infty} \setminus E^0$ by $\sigma(x) = x_{[2,k]}$ for $x \in E^k$, $1 \leq k \leq \infty$. Then σ is a partial local homeomorphism. Moreover, σ restricts to a partial local homeomorphism $\sigma: \partial E \setminus E_{\text{sing}}^0 \rightarrow \partial E$.*

Definition 3.1.21. The *path groupoid* is the Deaconu-Renault groupoid $\mathcal{TG}_E := G(E^{\leq \infty}, \sigma)$ and the *boundary path groupoid* is the Deaconu-Renault groupoid $\mathcal{G}_E := G(\partial E, \sigma)$.

The following result was first proved by Katsura [Kat09] in the case where E^0 and E^1 are compact and r is surjective. Yeend [Yee06] proved the result in full generality. Our notation is closer to that of Kumjian and Li [KL17, Theorem 7.7].

Theorem 3.1.22 ([Yee06, Theorem 5.2]). *Let E be a topological graph. Then*

$$C^*(\mathcal{T}\mathcal{G}_E) \cong \mathcal{T}_{X_E} \quad \text{and} \quad C^*(\mathcal{G}_E) \cong \mathcal{O}_{X_E}.$$

3.2 | Towards a groupoid model for \mathcal{O}_{X_E} : the core

The task of finding a “natural” groupoid model for \mathcal{O}_{X_E} for a topological quiver E is far from straightforward. The only analogue we have is in the case where E is a topological graph. In this case the associated groupoid is the Deaconu-Renault groupoid $\mathcal{G}_E = G(\partial E, \sigma)$ of Section 3.1.4. There are a number of issues that need to be addressed when it comes to building a groupoid model for a general topological quiver.

The first issue is that since the set E_{sing}^0 of singular vertices now also contains branched vertices E_{br}^0 , we do not have a clear candidate for the unit space of such a groupoid. The second, and more subtle issue, is how to endow such a groupoid with a Haar system. Since a topological quiver comes equipped with an s -system, it would be expected that the Haar system on such a groupoid should have some relation to the s -system on the topological quiver.

Recall from Definition A.3.10 that the Toeplitz algebra \mathcal{T}_{X_E} and Cuntz-Pimsner algebra \mathcal{O}_{X_E} , both admit a strongly continuous action of \mathbb{T} called the *gauge action*. The fixed-point algebras with respect to the gauge action are denoted by $\mathcal{T}_{X_E}^{\mathbb{T}}$ and $\mathcal{O}_{X_E}^{\mathbb{T}}$ and referred to as *cores*. In order to simplify matters, we instead focus on finding a groupoid model for the cores. The notion of core we are considering is different to the dynamical cores for topological quivers considered by Brenken [Bre10a].

When E is a topological graph, both the gauge action on $\mathcal{O}_{X_E} \cong C^*(\mathcal{G}_E)$ and the core admit descriptions in terms of the underlying groupoid. Let $c: \mathcal{G}_E \rightarrow \mathbb{Z}$ denote the map $c(x, k, y) = k$. Then [Ren80, Proposition II.5.1] implies that c induces a strongly continuous action $\beta: \mathbb{T} \rightarrow C^*(\mathcal{G}_E)$ satisfying $\beta_z(f)(x, k, y) = z^k f(x, k, y)$ for all $f \in C_c(\mathcal{G}_E)$, which fixes $C_0(\mathcal{G}_E^{(0)})$. The fixed-point algebra $C^*(\mathcal{G}_E)^{\beta}$ can be identified with the groupoid C^* -algebra of the closed subgroupoid

$$\mathcal{R}_E := \{(x, y) \in \mathcal{G}_E \mid \exists k \in \mathbb{N}_0 \text{ such that } x, y \in \text{dom}(\sigma^k), \sigma^k(x) = \sigma^k(y)\} \quad (3.4)$$

of \mathcal{G}_E . It follows from the proof of [Yee06, Theorem 5.2] that the isomorphism between \mathcal{O}_{X_E} and $C^*(\mathcal{G}_E)$ is \mathbb{T} -equivariant with respect to the gauge action and β . Consequently, the core $\mathcal{O}_{X_E}^{\mathbb{T}}$ is isomorphic to $C^*(\mathcal{R}_E)$. Similar considerations show that $\mathcal{T}_{X_E}^{\mathbb{T}}$ is isomorphic to the C^* -algebra of the subgroupoid

$$\mathcal{TR}_E := \{(x, y) \in \mathcal{T}\mathcal{G}_E \mid \exists k \in \mathbb{N}_0 \text{ such that } x, y \in \text{dom}(\sigma^k), \sigma^k(x) = \sigma^k(y)\} \quad (3.5)$$

of $\mathcal{T}\mathcal{G}_E$.

Returning to the case where E is a topological quiver, our goal is to find an analogue of \mathcal{R}_E and \mathcal{TR}_E that recovers $\mathcal{O}_{X_E}^{\mathbb{T}}$ and $\mathcal{T}_{X_E}^{\mathbb{T}}$ for a general topological quiver E .

The strategy is to decompose the core of \mathcal{O}_{X_E} and \mathcal{T}_{X_E} as a direct limits of C^* -algebras, and find groupoid models for each C^* -algebra in the corresponding directed system. By then constructing an appropriate limit of groupoids, it should be possible to arrive at a groupoid model for the core.

So far we have been unable to complete this programme for a general topological quiver. However, it can be completed for a topological graph, which we do in Section 3.3. Although we have limited success with topological quivers, the success in topological graph case indicates that with suitable modifications our strategy may still be feasible.

3.2.1 Decomposing the core

In [Kat04b, §5], Katsura examined the cores of C^* -algebras generated by representations of C^* -correspondences. In this section we outline Katsura's decomposition of the core as a direct limit. This decomposition is analogous to the decomposition of the AF -core of a Cuntz-Krieger algebra as a direct limit of finite dimensional C^* -algebras.

In Proposition 3.2.5, we give a characterisation of when a direct limit of C^* -algebras is isomorphic to the core of a C^* -algebra generated by a representation of a C^* -correspondence. Using this characterisation allows us to find groupoid models for each C^* -algebra in the directed system.

For the remainder of this subsection fix an arbitrary C^* -correspondence (ϕ, X) over a C^* -algebra A . Recall from Appendix A.2 that if (π, ψ) is a representation of X then for each $n \in \mathbb{N}$, then ψ induces representations (π, ψ^n) , where $\psi^n: X^{\otimes n} \rightarrow C^*(\pi, \psi)$ is such that $\psi^n(\xi_1 \otimes \cdots \otimes \xi_n) = \psi(\xi_1) \cdots \psi(\xi_n)$ for all $\xi_1, \dots, \xi_n \in X$. Moreover, ψ induces $*$ -homomorphisms $\psi^{(n)}: \text{End}_A^0(X^{\otimes n}) \rightarrow C^*(\pi, \psi)$ satisfying $\psi^{(n)}(\Theta_{\xi, \eta}) = \psi^n(\xi)\psi^n(\eta)^*$ for all $\xi, \eta \in X^{\otimes n}$. If π is injective then both ψ^n and $\psi^{(n)}$ are isometric.

Let $(j_A, j_X): (\phi, X) \rightarrow \mathcal{T}_X$ and $(i_A, i_X): (\phi, X) \rightarrow \mathcal{O}_X$ denote the universal representations of (ϕ, X) . Both $j_A: A \rightarrow \mathcal{T}_X$ and $i_A: A \rightarrow \mathcal{O}_X$ are injective by [Kat04b, Proposition 4.3] and [Kat04b, Proposition 4.9]. Following Katsura [Kat04b], for each $n \in \mathbb{N}_0$ define C^* -subalgebras $B_{[0,n]}$ of \mathcal{T}_X and $C_{[0,n]}$ of \mathcal{O}_X by,

$$\begin{aligned} B_{[0,n]} &:= j_A(A) + j_X^{(1)}(\text{End}_A^0(X)) + \cdots + j_X^{(n)}(\text{End}_A^0(X^{\otimes n})) \subseteq \mathcal{T}_X; \text{ and} \\ C_{[0,n]} &:= i_A(A) + i_X^{(1)}(\text{End}_A^0(X)) + \cdots + i_X^{(n)}(\text{End}_A^0(X^{\otimes n})) \subseteq \mathcal{O}_X. \end{aligned} \quad (3.6)$$

It follows from [Kat04b, Lemma 5.4] that $B_{[0,n]}$ and $C_{[0,n]}$ are indeed C^* -subalgebras. For each $n \in \mathbb{N}$ let $\iota_n: B_{[0,n]} \rightarrow B_{[0,n+1]}$ denote the natural inclusion. Abusing notation we also let ι_n denote the $\iota_n: C_{[0,n]} \rightarrow C_{[0,n+1]}$. The algebras $B_{[0,n]}$ and $C_{[0,n]}$ are used to give a direct limit decomposition of \mathcal{T}_X and \mathcal{O}_X , respectively.

Proposition 3.2.1 ([Kat04b, Proposition 5.7]). *If (ϕ, X) is a C^* -correspondence, then*

$$\mathcal{T}_X^\mathbb{T} \cong \varinjlim (B_{[0,n]}, \iota_n) \quad \text{and} \quad \mathcal{O}_X^\mathbb{T} \cong \varinjlim (C_{[0,n]}, \iota_n).$$

In general, if $(\pi, \psi): (\phi, X) \rightarrow D$ is a representation in a C^* -algebra D which admits a gauge action, we can define

$$B_{[0,n]}^{(\pi, \psi)} := \pi(A) + \psi^{(1)}(\text{End}_A^0(X)) + \cdots + \psi^{(n)}(\text{End}_A^0(X^{\otimes n})) \subseteq C^*(\pi, \psi).$$

It then follows from [Kat04b, Proposition 5.7] that $C^*(\pi, \psi)^\mathbb{T} \cong \varinjlim (B_{[0,n]}^{(\pi, \psi)}, \iota_n)$.

Although the algebras $B_{[0,n]}$ and $C_{[0,n]}$ allow us to describe $\mathcal{T}_X^\mathbb{T}$ and $\mathcal{O}_X^\mathbb{T}$ fairly effectively, the algebraic structure of $B_{[0,n]}$ and $C_{[0,n]}$ is unwieldy. Since both algebras occur as sums of algebras inside an ambient algebra, there are many identifications taking place. For example, in \mathcal{O}_{X_E} , Cuntz-Pimsner covariance implies that $i_A(a) = i_X^{(1)}(\phi(a))$ for all $a \in I_X$. Our immediate goal is to get a better handle on the algebraic structure of $B_{[0,n]}$ and $C_{[0,n]}$.

Recall from Appendix A.2 that for all $m \leq n$ and $T \in \text{End}_A(X^{\otimes m})$ there is an adjointable operator $T \otimes \text{id}_{m-n}$ in $\text{End}_A(X^{\otimes n})$ satisfying $(T \otimes \text{id}_{m-n})(\xi \otimes \eta) = T\xi \otimes \eta$ for all $\xi \in X^{\otimes m}$ and $\eta \in X^{\otimes n-m}$. The following lemma describes the product in both $B_{[0,n]}$ and $C_{[0,n]}$.

Lemma 3.2.2 ([Kat04b, Lemma 5.4]). *Let $m, n \in \mathbb{N}$, and suppose that $(\pi, \psi): (\phi, X) \rightarrow D$ is a representation of a C^* -correspondence (ϕ, X) over A . Then for all $T \in \text{End}_A^0(X^{\otimes m})$ and $S \in \text{End}_A^0(X^{\otimes n})$,*

$$\psi^{(m)}(T)\psi^{(n)}(S) = \begin{cases} \psi^{(m)}(T(S \otimes \text{id}_{m-n})) & \text{if } m \geq n; \\ \psi^{(n)}((T \otimes \text{id}_{n-m})S) & \text{if } n \geq m. \end{cases}$$

Recall from Definition A.3.8 that for a representation (π, ψ) of a C^* -correspondence (ϕ, X) , the ideal of covariance is defined by $I_{(\pi, \psi)} := \{a \in A \mid \pi(a) \in \psi^{(1)}(\text{End}_A^0(X))\}$. Katsura proves the following result.

Proposition 3.2.3 ([Kat04b, Proposition 5.12]). *Let $(\pi, \psi): (\phi, X) \rightarrow D$ be a representation of a C^* -correspondence (ϕ, X) over A , and let $I_{(\pi, \psi)}$ be the ideal of covariance. Then for each $n \in \mathbb{N}_0$ we have the following commutative diagram with exact rows:*

$$\begin{array}{ccccccc} 0 \longrightarrow \text{End}_A^0(X^{\otimes n} \cdot I_{(\pi, \psi)}) & \xrightarrow{\psi^{(n)}} & B_{[0,n]}^{(\pi, \psi)} & \longrightarrow & B_{[0,n]}^{(\pi, \psi)} / \psi^{(n)}(\text{End}_A^0(X^{\otimes n} \cdot I_{(\pi, \psi)})) & \longrightarrow & 0 \\ \downarrow T \mapsto T \otimes \text{id} & & \downarrow \iota_n & & \downarrow \cong & & \\ 0 \longrightarrow \text{End}_A^0(X^{\otimes n+1}) & \xrightarrow{\psi^{(n+1)}} & B_{[0,n+1]}^{(\pi, \psi)} & \longrightarrow & B_{[0,n]}^{(\pi, \psi)} / \psi^{(n+1)}(\text{End}_A^0(X^{\otimes n+1})) & \longrightarrow & 0 \end{array} \quad (3.7)$$

Remark 3.2.4. The ideal of covariance for (j_A, j_X) is $\{0\}$ and the ideal of covariance for (i_A, i_X) is the Katsura ideal I_X . In particular, the top row of the diagram (3.7) implies that $B_{[0,n]} \cong B_{[0,n]} / j_X^{(n+1)}(\text{End}_A^0(X^{\otimes n+1}))$. Thus, for the universal representation (j_A, j_X) , the diagram (3.7) becomes the split exact sequence

$$0 \longrightarrow \text{End}_A^0(X^{\otimes n+1}) \xrightarrow{j_X^{(n+1)}} B_{[0,n+1]} \xrightarrow{\quad \xleftarrow{\iota_n} \quad} B_{[0,n]} \longrightarrow 0. \quad (3.8)$$

In light of Proposition D.0.1 the $*$ -linear structure of both $B_{[0,n]}$ and $C_{[0,n]}$ can be determined inductively using Proposition 3.2.3. Together with Lemma 3.2.2, these two results completely characterise the algebras $B_{[0,n]}$ and $C_{[0,n]}$.

Proposition 3.2.5. *Let $(\pi, \psi): (\phi, X) \rightarrow D$ be a representation of a C^* -correspondence (ϕ, X) over A , and let $I_{(\pi, \psi)}$ be the ideal of covariance. Suppose that $(A_n, \tau_n: A_n \rightarrow A_{n+1})_{n \in \mathbb{N}_0}$ is a directed system of C^* -algebras with the following properties:*

(i) there are isomorphisms α_n from $\text{End}_A^0(X^{\otimes n})$ onto ideals $\alpha_n(\text{End}_A^0(X^{\otimes n})) \triangleleft A_n$, with $\alpha_0(A) = A_0$;

(ii) for each $n \in \mathbb{N}_0$ there is a commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{End}_A^0(X^{\otimes n} \cdot I_{(\pi, \psi)}) & \xrightarrow{\alpha_n} & A_n & \longrightarrow & A_n / \alpha_n(\text{End}_A^0(X^{\otimes n} \cdot I_{(\pi, \psi)})) \longrightarrow 0 \\ & & \downarrow T \mapsto T \otimes \text{id} & & \downarrow \tau_n & & \downarrow \cong \\ 0 & \longrightarrow & \text{End}_A^0(X^{\otimes n+1}) & \xrightarrow{\alpha_{n+1}} & A_{n+1} & \longrightarrow & A_{n+1} / \alpha_{n+1}(\text{End}_A^0(X^{\otimes n+1})) \longrightarrow 0 \end{array}$$

with exact rows; and

(iii) with $\tau_{[n, m]} := \tau_{m-1} \circ \cdots \circ \tau_n$, for all $T \in \text{End}_A^0(X^{\otimes m})$ and $S \in \text{End}_A^0(X^{\otimes n})$ we have

$$\alpha_m(T) \tau_{[n, m]}(\alpha_n(S)) = \alpha_m(T(S \otimes \text{id}_{m-n}))$$

for $m > n$, and

$$\tau_{[m, n]}(\alpha_m(T)) \alpha_n(S) = \alpha_n((T \otimes \text{id}_{n-m})S)$$

for $n > m$.

Then $A_n \cong B_{[0, n]}^{(\pi, \psi)}$ for all $n \in \mathbb{N}$. In particular, if (π, ψ) admits a gauge action, then $\varinjlim (A_n, \tau_n) \cong C^*(\pi, \psi)^{\mathbb{T}}$.

Proof. For induction, suppose that there is an isomorphism $\Upsilon_n: B_{[0, n]}^{(\pi, \psi)} \rightarrow A_n$ and that the isomorphism satisfies $\alpha_n = \Upsilon_n \circ \psi^{(n)}$. This is true for $n = 0$ since α_0 is an isomorphism by hypothesis. Since $\alpha_n = \Upsilon_n \circ \psi^{(n)}$, it follows that Υ_n restricts to an isomorphism between $B_{[0, n]}^{(\pi, \psi)} / \psi^{(n)}(\text{End}_A^0(X^{\otimes n} \cdot I_{(\pi, \psi)}))$ and $A_n / \alpha_n(\text{End}_A^0(X^{\otimes n} \cdot I_{(\pi, \psi)}))$. Consequently, the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{End}_A^0(X^{\otimes n} \cdot I_{(\pi, \psi)}) & \xrightarrow{\psi^{(n)}} & B_{[0, n]}^{(\pi, \psi)} & \longrightarrow & B_{[0, n]}^{(\pi, \psi)} / \psi^{(n)}(\text{End}_A^0(X^{\otimes n} \cdot I_{(\pi, \psi)})) \longrightarrow 0 \\ & & \downarrow T \mapsto T \otimes \text{id} & & \downarrow \tau_n \circ \Upsilon_n & & \downarrow \cong \\ 0 & \longrightarrow & \text{End}_A^0(X^{\otimes n+1}) & \xrightarrow{\alpha_{n+1}} & A_{n+1} & \longrightarrow & A_{n+1} / \alpha_{n+1}(\text{End}_A^0(X^{\otimes n+1})) \longrightarrow 0 \end{array}$$

commutes and has exact rows. Proposition D.0.1 and Proposition 3.2.3 imply that there is a $*$ -preserving isomorphism of vector spaces $\Upsilon_{n+1}: B_{[0, n+1]}^{(\pi, \psi)} \rightarrow A_{n+1}$ satisfying

$$\Upsilon_{n+1}(\psi^{(n+1)}(T) + b) = \alpha_{n+1}(T) + \tau_n \circ \Upsilon_n(b)$$

for all $T \in \text{End}_A^0(X^{\otimes n+1})$ and $b \in B_{[0, n]}^{(\pi, \psi)}$. In particular, $\Upsilon_{n+1} \circ \psi^{(n+1)} = \alpha_{n+1}$.

Inductively, we see that each $a \in A_{n+1}$ can be written in the form $a = \sum_{k=0}^{n+1} \tau_{[k, n+1]} \circ \alpha_k(T_k)$ for some $T_k \in \text{End}_A^0(X^{\otimes k})$. It follows that,

$$\Upsilon_{n+1} \left(\sum_{k=0}^{n+1} \psi^{(k)}(T_k) \right) = \sum_{k=0}^{n+1} \tau_{[k, n+1]} \circ \alpha_k(T_k).$$

Thus, for multiplicativity of Υ_{n+1} it suffices to show that

$$\Upsilon_{n+1}(\psi^{(k)}(T)\psi^{(l)}(S)) = \Upsilon_{n+1}(\psi^{(k)}(T))\Upsilon_{n+1}(\psi^{(l)}(S)),$$

for all $T \in \text{End}_A^0(X^{\otimes k})$ and $S \in \text{End}_A^0(X^{\otimes l})$, $0 \leq k, l \leq n$. This is immediate from (iii). Automatic continuity of C^* -homomorphisms now gives the result. \square

Returning to the case where $X = X_E$ for a topological quiver E , Proposition 3.2.1 gives us a strategy for determining a groupoid model for $\mathcal{O}_{X_E}^{\mathbb{T}}$ and $\mathcal{T}_{X_E}^{\mathbb{T}}$. If we can find groupoid models for $B_{[0,n]}$ and $C_{[0,n]}$, and then perform a limiting operation on the associated groupoids, we should be able to recover groupoid models for the cores. In order to find groupoid models for $B_{[0,n]}$ and $C_{[0,n]}$ we must find a groupoid model for $\text{End}_A^0(X_E^{\otimes n})$ and a way to “glue” the resulting groupoids together in order to perform the internal sum of (3.6) on the level of C^* -algebras. Finally we verify that the C^* -algebras associated to the groupoids we construct satisfy the hypotheses of Proposition 3.2.5. Summarising, our strategy is:

- (i) Find a groupoid model for $\text{End}_A^0(X_E^{\otimes n})$ for each $n \in \mathbb{N}$.
- (ii) Implement the map $T \mapsto T \otimes \text{id}$ of (3.7) on the level of groupoids for $I_{(\pi, \psi)} = I_{X_E}$ (Recall that $I_{(j_A, j_{X_E})} = \{0\}$ in the Toeplitz case).
- (iii) Inductively construct groupoids G_n such that $C^*(G_n) \cong B_{[0,n]}$, by starting with $G_0 = E^0$ and “gluing” G_n to the groupoid associated to $\text{End}_A^0(X_E^{\otimes n+1})$ in such a way that the hypotheses of Proposition 3.2.5 are satisfied. Perform a similar process for $C_{[0,n]}$.
- (iv) Take a suitable limit of the groupoids G_n in order to obtain a groupoid model for $\mathcal{T}_{X_E}^{\mathbb{T}}$. Perform a similar process for $\mathcal{O}_{X_E}^{\mathbb{T}}$.

As mentioned earlier, this strategy can be implemented completely in the case where E is a topological graph, with the resulting limit groupoids being isomorphic to \mathcal{R}_E and \mathcal{TR}_E . However, once we enter the non-étale realm of branched points, obstructions to this strategy show up during the “gluing” step. Although the gluing can be achieved at a topological level, the existence of a Haar system becomes problematic. Precisely what goes wrong is explained in Remark 3.3.18 and Remark 3.3.22.

3.2.2 A groupoid model for $\text{End}_A^0(X_E)$

Let $E = (E^0, E^1, r, s, \lambda)$ be a topological quiver. We begin the first step in our programme by constructing a groupoid model for the compact operators on X_E . A notable feature of the groupoid is that it is typically non-étale.

Definition 3.2.6. Let $E = (E^0, E^1, r, s, \lambda)$ be a topological quiver and define an equivalence relation on E^1 by,

$$R := E^1 \times_{s,s} E^1 = \{(x, y) \in E^1 \times E^1 \mid s(x) = s(y)\}.$$

Equip R with the subspace topology inherited from $E^1 \times E^1$. If $(x, y) \in R$ we say that x and y are *s-equivalent*.

The equivalence relation R defines a locally compact Hausdorff groupoid: elements $(x, y), (z, w) \in R$ are composable if and only if $y = z$, in which case the product is given by $(x, y)(y, w) = (x, w)$. Inverses are given by $(x, y)^{-1} = (y, x)$. The groupoid R has unit space $R^{(0)} = \{(x, x) \in E^1 \times E^1\}$ which we usually identify with E^1 . To avoid confusion with the range and source maps in E , we write $r_R : R \rightarrow R^{(0)}$ and $s_R : R \rightarrow R^{(0)}$ for the range and source maps of the groupoid R , when required. In particular, $r_R(x, y) = (x, x)$ and $s_R(x, y) = (y, y)$. Both of r_R and s_R are both open because s is open. Since s is continuous R is closed in $E^1 \times E^1$.

The unit space $R^{(0)}$ is closed in R as E^1 is Hausdorff. However, $R^{(0)}$ is not always open in R . Equivalently, the groupoid R is not always étale. The failure of R to be étale can be characterised by the failure of the source map $s : E^1 \rightarrow E^0$ to be locally injective.

Lemma 3.2.7. *Let $E = (E^0, E^1, r, s, \lambda)$ be a topological quiver and let $x \in E^1$. Then the following are equivalent:*

- (i) *there is an open neighbourhood $U \subseteq E^1$ of x such that $s|_U$ is injective; and*
- (ii) *there is an open neighbourhood $V \subseteq R$ of (x, x) such that $V \subseteq R^{(0)}$.*

Proof. For (i) \implies (ii) let $V = (U \times U) \cap R$ and note that if $(y, z) \in V$ then $y, z \in U$ and $s(y) = s(z)$. Injectivity implies $y = z$, so $(y, z) \in R^{(0)}$. For (ii) \implies (i) choose a basic open neighbourhood $(U \times U) \cap R$ of (x, x) contained in $V \subseteq R^{(0)}$. If $y, z \in U$ are such that $s(y) = s(z)$, then $(y, z) \in (U \times U) \cap R \subseteq V \subseteq R^{(0)}$. Hence, $y = z$. \square

The following shows that R need not be étale.

Example 3.2.8. Let $([0, 1], \Gamma' = \{\gamma_1, \gamma_2\})$ be the iterated function system of Example 2.2.3. Identifying E^1 with $[0, 1]$ via the range map $r : E^1 \rightarrow [0, 1]$, the relation R is given by

$$R = \{(x, x) \mid x \in [0, 1]\} \cup \{(x, 1 - x) \mid x \in [0, 1]\} \subseteq [0, 1] \times [0, 1].$$

The graph of R is shown in Figure 3.1. For each $(x, x) \in R^{(0)}$ we have

$$R^{(x, x)} = \begin{cases} \{(x, x), (x, 1 - x)\} & \text{if } x \neq 1/2; \text{ and} \\ \{(x, x)\} & \text{if } x = 1/2. \end{cases}$$

In both cases, $R^{(x, x)}$ is discrete. However, R is not r -discrete as any open neighbourhood of $(1/2, 1/2)$ contains elements of $R \setminus R^{(0)}$. \blacktriangle

Although R is not étale in general, the s -system λ on E^1 can still be used to define a Haar system on R . For each $x \in R^{(0)}$ define $\mu^x := \delta_x \times \lambda_{s(x)}$ as a measure on R . Then μ^x is a σ -finite Radon measure, and for all $f \in C_c(R)$

$$\int_{R^x} f(x, y) d\mu^x(x, y) = \int_{s^{-1}(s(x))} f(x, y) d\lambda_{s(x)}(y). \quad (3.9)$$

Since λ_v has support $s^{-1}(v)$ for each $v \in E^0$, the measure μ^x has support R^x for each $x \in R^{(0)}$. Similarly, we define $\mu_x = \lambda_{s(x)} \times \delta_x$, which has support R_x . The following is an immediate consequence of the definition of μ_x (cf. Lemma 3.1.15).

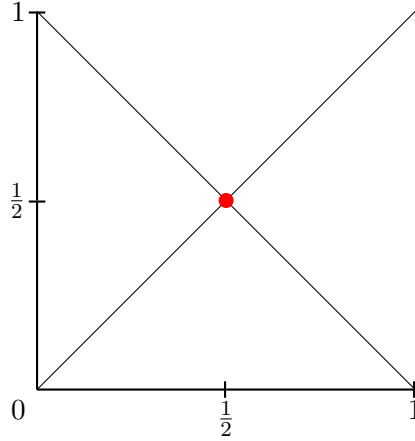


Figure 3.1: The graph of the relation R of Example 3.2.8 with the point $(1/2, 1/2)$ highlighted.

Lemma 3.2.9. *For each $x \in R^{(0)} \simeq E^1$ there is a unitary $U_x: L^2(R_x, \mu_x) \rightarrow \mathcal{H}_{s(x)} := L^2(s^{-1}(s(x)), \lambda_{s(x)})$ satisfying $U_x(\xi)(z) = \xi(z, x)$ for all $\xi \in C_c(R_x)$.*

As promised, we have the following.

Proposition 3.2.10. *The collection $\mu := \{\mu^x\}_{x \in R^{(0)}}$ defines a Haar system on R .*

Proof. We have $\text{supp}(\mu^x) = r_R^{-1}(x, x)$. For invariance note that if $(x, y) \in R$ then $s(y) = s(x)$. In particular, $\lambda_{s(x)} = \lambda_{s(y)}$, and invariance follows. That $x \mapsto \mu^x(f)$ is continuous for each $f \in C_c(R)$ follows from (i) of Definition 3.1.1. \square

We can now construct a groupoid C^* -algebra $C^*(R)$ by completing the convolution algebra $C_c(R)$. The following amenability result implies that the completion is unique.

Lemma 3.2.11. *The groupoid R is topologically amenable. In particular, the full and reduced norms on $C_c(R)$ agree.*

Proof. The groupoid R is a groupoid bundle over E^0 (see Definition B.3.3). To see why, define $p: R^{(0)} \rightarrow E^0$ by $p(x, x) = s(x)$. The definition of R implies that, $p \circ r_R = p \circ s_R$. Moreover, p is open since s is open. So it suffices to show that $R(v) := R|_{p^{-1}(v)}$ is amenable for all $v \in E^0$: the result then follows from Theorem B.3.4 and Definition B.3.1.

For each $v \in E^0$, $R(v)$ is the full equivalence relation on the space $s^{-1}(v)$. Let e_n be a sequence of positive functions in $C_c(s^{-1}(v))$ such that e_n converges uniformly to 1 on compacta in $s^{-1}(v)$. Suppose that $g \in C_c(s^{-1}(v))$ is a positive function such that $\|g\|_{L^2(s^{-1}(v), \lambda_v)} = 1$, and let $f_n \in C_c(s^{-1}(v))$ be given by $f_n(x, y) = e_n(x)g(y)$. Then,

$$(f_n * f_n^*)(x, y) = \int f_n(x, z) \overline{f_n(y, z)} d\mu^x(x, z) = e_n(x)e_n(y) \int |g(z)|^2 d\lambda_{s(x)}(z) = e_n(x)e_n(y).$$

Hence, $f_n * f_n^*$ converges uniformly on compacta to 1. \square

The aim for the remainder of this subsection is to show that the groupoid algebra $C^*(R)$ is isomorphic to the algebra $\text{End}_A^0(X_E)$ of compact operators on X_E . Before we begin, we require a technical lemma to give us some control over the support of functions in $C_c(R)$.

Lemma 3.2.12. *Let Y be a second-countable locally compact space, and suppose that R is a closed relation on Y . Then there exist collections $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ of compact subsets of Y satisfying:*

- (i) $A_n \subseteq \text{int}(A_{n+1})$ for each $n \in \mathbb{N}$;
- (ii) $B_n \subseteq \text{int}(B_{n+1})$ for each $n \in \mathbb{N}$;
- (iii) $R = \bigcup_{n \in \mathbb{N}} (A_n \times B_n) \cap R$;
- (iv) $(A_n \times B_n) \cap R \subseteq \text{int}_R((A_{n+1} \times B_{n+1}) \cap R)$ for each $n \in \mathbb{N}$; and
- (v) $\text{int}_R((A_n \times B_n) \cap R) = (\text{int}(A_n) \times \text{int}(B_n)) \cap R$ for each $n \in \mathbb{N}$.

Here, int_R denotes the interior relative to R .

Proof. We regard R as a groupoid in the usual way with $r_R: R \rightarrow Y$ and $s_R: R \rightarrow Y$ being the range and source maps, given by $r_R(x, y) = x$ and $s_R(x, y) = y$. The relation R is second-countable, locally compact, and Hausdorff since R is closed in $Y \times Y$, so R is σ -compact. Accordingly, there is a compact exhaustion $\{K_n\}_{n \in \mathbb{N}}$ of R . That is, each K_n is compact, $R = \bigcup_{n \in \mathbb{N}} K_n$, and $K_n \subseteq \text{int}(K_{n+1})$ for all $n \in \mathbb{N}$.

Let $A_1 = r_R(K_1)$ and $B_1 = s_R(K_1)$. Since R is closed, $(A_1 \times B_1) \cap R$ is a compact subset of R . In particular, there exists $n_2 \in \mathbb{N}$ such that $(A_1 \times B_1) \cap R \subseteq \text{int}(K_{n_2})$. Let $A_2 = r_R(K_{n_2})$ and $B_2 = s_R(K_{n_2})$. Since r_R and s_R are open it follows that $A_1 \subseteq \text{int}(A_2)$ and $B_1 \subseteq \text{int}(B_2)$. Since $K_{n_2} \subseteq (A_2 \times B_2) \cap R$ it follows that $(A_1 \times B_1) \cap R \subseteq \text{int}((A_2 \times B_2) \cap R)$. Now inductively define, for each $n_k \in \mathbb{N}$, $A_k = r_R(K_{n_k})$, and $B_k = s_R(K_{n_k})$. Then (iii) is satisfied since $K_{n_k} \subseteq (A_k \times B_k) \cap R$.

To see that (v) holds note that $\text{int}(A_n) \times \text{int}(B_n) = \text{int}_{E^1 \times E^1}(A_n \times B_n)$. It follows that $(\text{int}(A_n) \times \text{int}(B_n)) \cap R$ is open in the relative topology of R and contained in $(A_n \times B_n) \cap R$. For the reverse inclusion note that since r_R is open, the set $r_R(\text{int}((A_n \times B_n) \cap R))$ is open in A_n , and hence open as subset of $\text{int}(A_n)$. Similarly, $s_R(\text{int}((A_n \times B_n) \cap R)) \subseteq \text{int}(B_n)$. Thus, if $(x, y) \in \text{int}((A_n \times B_n) \cap R)$ we have $(x, y) \in R$, $x \in \text{int}(A_n)$ and $y \in \text{int}(B_n)$. \square

Using Lemma 3.2.12 we can approximate functions in $C_c(R) \subseteq C^*(R)$ using products of functions in $C_c(E^1)$.

Lemma 3.2.13. *For each $\xi, \eta \in C_c(E^1)$ let $f_{\xi, \eta}$ denote the function on R given by $f_{\xi, \eta}(x, y) = \xi(x)\eta(y)$ for all $(x, y) \in R$. Then $C^*(R) = \overline{\text{span}}\{f_{\xi, \eta} \mid \xi, \eta \in C_c(E^1)\}$.*

Proof. Fix $g \in C_c(R)$. Let $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ be collections of compact sets as in Lemma 3.2.12. Then there exists $n \in \mathbb{N}$ such that $\text{supp}(g) \subseteq \text{int}((A_n \times B_n) \cap R)$. In particular, $g \in C_0(\text{int}((A_n \times B_n) \cap R)) \subseteq C_0(R)$. We write $A := A_n$ and $B := B_n$ for such an n .

Let $\xi \in C_c(\text{int}(A))$ and $\eta \in C_c(\text{int}(B))$, where $C_c(\text{int}(A))$ and $C_c(\text{int}(B))$ are thought of as $*$ -subalgebras of $C_c(E^1)$. Consider the function $h_{\xi, \eta}$ on $E^1 \times E^1$ given by $h_{\xi, \eta}(x, y) = \xi(x)\eta(y)$. Then $h_{\xi, \eta}$ is continuous on $E^1 \times E^1$ and compactly supported with $\text{supp}(h_{\xi, \eta}) \subseteq \text{supp}(\xi) \times \text{supp}(\eta)$. The function $f_{\xi, \eta}$ is given as the restriction of $h_{\xi, \eta}$ to R . In particular, $f_{\xi, \eta}$ is continuous and $\text{supp}(f_{\xi, \eta}) \subseteq (\text{supp}(\xi) \times \text{supp}(\eta)) \cap R \subseteq \text{int}_R((A \times B) \cap R)$ by (v) of Lemma 3.2.12.

We aim to use the Stone-Weierstrass Theorem to see that the $*$ -algebra generated by $Z := \{f_{\xi, \eta} \mid \xi \in C_c(\text{int}(A)), \eta \in C_c(\text{int}(B))\}$ is dense in $C_0(\text{int}((A \times B) \cap R))$ in the uniform norm. Fix $(x, y) \in \text{int}_R((A \times B) \cap R) = (\text{int}(A) \times \text{int}(B)) \cap R$. Since $C_c(\text{int}(A))$ and

$C_c(\text{int}(B))$ vanish nowhere in $\text{int}(A)$ and $\text{int}(B)$ respectively, there exist $\xi \in C_c(\text{int}(A))$ and $\eta \in C_c(\text{int}(B))$ such that $\xi(x) \neq 0$ and $\eta(y) \neq 0$. It follows that $f_{\xi,\eta}(x, y) \neq 0$. Thus, Z vanishes nowhere in $\text{int}((A \times B) \cap R)$. A similar argument shows that Z separates points. Consequently, the $*$ -algebra generated by Z is dense in $C_0(\text{int}((A \times B) \cap R))$, in the uniform norm by the Stone-Weierstrass Theorem.

Since Z generates a dense $*$ -subalgebra of $C_0(\text{int}((A \times B) \cap R))$, for each $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$, $\xi_{i,k} \in C_c(\text{int}(A))$, and $\eta_{i,k} \in C_c(\text{int}(B))$ such that $\|g - \sum_{i=1}^{N_k} f_{\xi_{i,k}, \eta_{i,k}}\|_\infty < \frac{1}{k}$. Let $a_k = \sum_{i=1}^{N_k} f_{\xi_{i,k}, \eta_{i,k}}$. Since $\text{supp}(a_k) \subseteq (A \times B) \cap R$ for all $k \in \mathbb{N}$, it follows from [Wil19, Lemma C.1.] that $a_k \rightarrow g$ in the inductive limit topology on $C_c(E^1)$. The I -norm on $C_c(R)$ defines a coarser topology than the inductive limit topology [Ren80, Proposition 1.4], so $\|g - a_k\|_I \rightarrow 0$. Since groupoid representations are I -norm bounded it follows that $\|g - a_k\|_r \rightarrow 0$. Since $C_c(R)$ is dense in $C^*(R)$, the result follows. \square

We now come to the main result of this section: the isomorphism between $\text{End}_A^0(X_E)$ and $C^*(R)$. Under this isomorphism functions $f \in C_c(R)$ act as the kernels of integral-type operators on X_E .

Proposition 3.2.14. *There is an isomorphism $\Psi : \text{End}_A^0(X_E) \rightarrow C^*(R)$ satisfying*

$$\Psi(\Theta_{\xi,\eta})(x, y) = \xi(x) \overline{\eta(y)} \quad (3.10)$$

for all $\xi, \eta \in C_c(E^1)$ and $(x, y) \in R$. The inverse $\Phi : C^*(R) \rightarrow \text{End}_A^0(X_E)$ satisfies

$$\Phi(g)(\xi)(x) = \int_{E^1} g(x, y) \xi(y) d\lambda_{s(x)}(y) \quad (3.11)$$

for all $g \in C_c(R)$, $\xi \in C_c(E^1)$ and $x \in E^1$.

Proof. Since R is topologically amenable it suffices to consider the reduced C^* -algebra $C_r^*(R)$. Let $\text{End}_{A,c}^{00}(X_E) := \text{span}\{\Theta_{\xi,\eta} \mid \xi, \eta \in C_c(E^1)\}$. Define a map $\Psi : \text{End}_{A,c}^{00}(X_E) \rightarrow C_c(R)$ by linearly extending (3.10). For each $x \in R^{(0)} \simeq E^1$ let $\pi_x : C_c(R) \rightarrow B(L^2(R_x, \mu_x))$ denote the associated regular representation. We adopt the notation of Lemma 3.1.15 and Remark 3.1.16. Let $U_x : L^2(R_x, \mu_x) \rightarrow \mathcal{H}_{s(x)}$ be as in Lemma 3.2.9. Lemma 3.2.9 implies that for all $\xi, \eta \in C_c(E^1)$ and $h \in C_c(R_x)$ we have,

$$\begin{aligned} \pi_x(\Psi(\Theta_{\xi,\eta}))(h)(y, x) &= \int_{R^y} \Psi(\Theta_{\xi,\eta})(y, z) h(z, x) d\mu^y(y, z) \\ &= \int_{R^y} \xi(y) \overline{\eta(z)} h(z, x) d\mu^y(y, z) \\ &= \int_{s^{-1}(s(x))} \xi|_{s(x)}(y) \overline{\eta|_{s(x)}(z)} U_x(h)(z) d\lambda_{s(x)}(z) \\ &= \xi|_{s(x)}(y) \cdot (\eta|_{s(x)} \mid U_x(h))_{\mathcal{H}_{s(x)}}. \end{aligned}$$

With $\Theta_{\xi|_{s(x)}, \eta|_{s(x)}}$ denoting a rank-1 operator on the Hilbert space $\mathcal{H}_{s(x)}$, it follows that $\pi_x(\Psi(\Theta_{\xi,\eta})) = U_x^* \Theta_{\xi|_{s(x)}, \eta|_{s(x)}} U_x$. Fix a finite rank operator $T = \sum_{i=1}^k \Theta_{\xi_i, \eta_i}$ with $\xi_i, \eta_i \in C_c(E^1)$. Then, $\pi_x(\Psi(T)) = \sum_{i=1}^k U_x^* \Theta_{\xi_i|_{s(x)}, \eta_i|_{s(x)}} U_x$. Recall from Remark 3.1.16 that under the identification of $\text{End}_A^0(X_E)$ with $C_0(E^1, \mathcal{K})$ the operator $T \in \text{End}_A^0(X_E)$ is identified with the unique element $T' \in C_0(E^1, \mathcal{K})$ which satisfies $(T'(v) f_\xi(v) \mid f_\eta(v))_{\mathbb{C}} = (T\xi \mid \eta)_A(v)$ for all $v \in E^0$ and $\xi, \eta \in C_c(E^1)$. Using this identification at the last equality, we have

$$\|\Psi(T)\|_r = \sup_{x \in R^{(0)}} \|\pi_x(\Psi(T))\| = \sup_{x \in R^{(0)}} \left\| \sum_{i=1}^k \Theta_{f_{\xi_i}, f_{\eta_i}}(s(x)) \right\| = \sup_{v \in E^0} \|T'(v)\| = \|T\|.$$

Thus, Ψ extends to an isometric linear embedding $\Psi: \text{End}_A^0(X_E) \rightarrow C_r^*(R)$. The map Ψ is clearly $*$ -preserving. Multiplicativity on the dense $*$ -subalgebra $\text{End}_{A,c}^{00}(X_E)$ follows from linearly extending the computation:

$$\begin{aligned} \Psi(\Theta_{\xi,\eta}\Theta_{\alpha,\beta})(x,y) &= \Psi(\Theta_{\xi \cdot (\eta|\alpha),\beta})(x,y) \\ &= \xi(x) \left(\int_{s^{-1}s(x)} \eta(z) \overline{\alpha(z)} d\lambda_{s(x)}(z) \right) \overline{\beta(y)} \\ &= \int_{R^x} \xi(x) \eta(z) \overline{\alpha(z)} \beta(y) d\mu^x(x,z) \\ &= \Psi(\Theta_{\xi,\eta}) * \Psi(\Theta_{\alpha,\beta})(x,y). \end{aligned}$$

Since Ψ is isometric, it extends to a $*$ -homomorphism $\Psi: \text{End}_A^0(X) \rightarrow C^*(R)$. Lemma 3.2.13 implies that Ψ is surjective, since $\Psi(\Theta_{\xi_i,\eta_i}) = f_{\xi_i,\overline{\eta_i}}$. Hence, Ψ is an isomorphism.

Although we now know that Ψ is an isomorphism, it is not clear that (3.11) defines the inverse Φ for Ψ . For each $g \in C_c(R)$ and $\xi \in C_c(E^1)$ define a function $\Phi(g)\xi: E^1 \rightarrow E^1$ by

$$\Phi(g)\xi(x) = \int_{E^1} g(x,y)\xi(y) d\lambda_{s(x)}(y).$$

Then $\text{supp}(\Phi(g)\xi) \subseteq s(\text{supp}(g))$, so $\Phi(g)\xi$ is compactly supported. We claim that $\Phi(g)\xi$ is continuous. Fix $x \in E^1$, $\varepsilon > 0$ and a sequence $x_i \rightarrow x$ in E^1 . Then

$$\begin{aligned} & \left| \int_{E^1} g(x,y)\xi(y) d\lambda_{s(x)}(y) - \int_{E^1} g(x_i,y')\xi(y') d\lambda_{s(x_i)}(y') \right| \\ & \leq \left| \int_{E^1} g(x,y)\xi(y) d\lambda_{s(x)}(y) - \int_{E^1} g(x,y')\xi(y') d\lambda_{s(x_i)}(y') \right| \\ & \quad + \left| \int_{E^1} g(x,y')\xi(y') d\lambda_{s(x_i)}(y') - \int_{E^1} g(x_i,y')\xi(y') d\lambda_{s(x_i)}(y') \right|. \end{aligned}$$

Since $y \mapsto g(x,y)\xi(y)$ is a continuous compactly supported function on E^1 , Definition 3.1.2 (ii) implies that $|\int_{E^1} g(x,y)\xi(y) d\lambda_{s(x)}(y) - \int_{E^1} g(x,y')\xi(y') d\lambda_{s(x_i)}(y')| < \varepsilon$ for large i . On the other hand, continuity of g implies that $|g(x,y') - g(x_i,y')| < \varepsilon$ for large i . Since $v \mapsto \int_{E^1} |\xi(y)| d\lambda_v(y)$ belongs to $C_c(E^0)$, for large i we have

$$\begin{aligned} & \left| \int_{E^1} g(x,y')\xi(y') d\lambda_{s(x_i)}(y') - \int_{E^1} g(x_i,y')\xi(y') d\lambda_{s(x_i)}(y') \right| \\ & \leq \int_{E^1} |g(x,y') - g(x_i,y')| |\xi(y')| d\lambda_{s(x_i)}(y') \\ & < \varepsilon \left\| v \mapsto \int_{E^1} |\xi(y)| d\lambda_v(y) \right\|_\infty. \end{aligned}$$

We have now shown that $\Phi(g)\xi \in C_c(E^1)$. We now claim that $\xi \mapsto \Phi(g)\xi$ on $C_c(E^1)$

extends to a linear operator on X_E . Using the Cauchy-Swartz inequality in \mathcal{H}_v on the fourth line we compute:

$$\begin{aligned}
\|\Phi(g)\xi\|^2 &= \sup_{v \in E^0} (\Phi(g)\xi \mid \Phi(g)\xi)_A(v) \\
&= \sup_{v \in E^0} \int_{E^1} \left| \int_{E^1} g(x, y) \xi(y) d\lambda_v(y) \right|^2 d\lambda_v(x) \\
&= \sup_{v \in E^0} \int_{E^1} |\overline{g(x, \cdot)} \mid \xi|_{\mathcal{H}_v}|^2 d\lambda_v(x) \\
&\leq \sup_{v \in E^0} \int_{E^1} \left(\int_{E^1} |g(x, y)|^2 d\lambda_v(y) \right) \left(\int_{E^1} |\xi(y)|^2 d\lambda_v(y) \right) d\lambda_v(x) \\
&\leq \|\xi\|^2 \sup_{v \in E^0} \int_{E^1} \int_{E^1} |g(x, y)|^2 d\lambda_v(y) d\lambda_v(x).
\end{aligned}$$

We claim that the function $\phi_v : x \mapsto \int_{E^1} |g(x, y)|^2 d\lambda_v(y)$ belongs to $C_c(E^1)$. The function ϕ_v is compactly supported with support contained in $s(\text{supp}(g))$. For continuity fix $x \in E^1$, $\varepsilon > 0$ and a sequence $x_i \rightarrow x$. By continuity of g , for large i we have

$$\left| \int_{E^1} |g(x, y)|^2 d\lambda_v(y) - \int_{E^1} |g(x_i, y)|^2 d\lambda_v(y) \right| \leq \int_{\text{supp}(g)} \varepsilon d\lambda_v(y) = \varepsilon \lambda_v(\text{supp}(g)).$$

Since λ_v is a Radon measure, $\lambda_v(\text{supp}(g))$ is finite. Hence, ϕ_v is continuous. Thus, Definition 3.1.2 (ii) implies that $v \mapsto \int_{E^1} \phi_v(x) d\lambda_v(x)$ belongs to $C_c(E^0)$, and therefore $v \mapsto \int_{E^1} \phi_v(x) d\lambda_v(x)$ has finite supremum norm. Consequently,

$$\|\Phi(g)\xi\|^2 \leq \|\xi\|^2 \sup_{v \in E^0} \left| \int_{E^1} \phi_v(x) d\lambda_v(x) \right|$$

In particular, $\Phi(g) : \xi \mapsto \Phi(g)\xi$ extends to a bounded linear operator on X_E . Moreover, $\Phi(g)$ is right A -linear and adjointable with adjoint satisfying

$$\Phi(g)^* \xi(x) = \int_{E^1} \overline{g(y, x)} \xi(y) d\lambda_{s(x)}(y)$$

for all $\xi \in C_c(E^1)$. Now, for all $\xi, \eta, \zeta \in C_c(E^1)$ we have

$$\begin{aligned}
(\Phi \circ \Psi)(\Theta_{\xi, \eta})\zeta(x) &= \int_{s^{-1}(s(x))} \Psi(\Theta_{\xi, \eta})(x, y) \zeta(y) d\lambda_{s(x)}(y) \\
&= \int_{s^{-1}(s(x))} \overline{\xi(x)} \eta(y) \zeta(y) d\lambda_{s(x)}(y) \\
&= \Theta_{\xi, \eta} \zeta(x).
\end{aligned}$$

Since Ψ maps $\text{End}_{A,c}^{00}(X_E)$ onto $C_c(R)$,

$$\begin{aligned} \|\Phi\| &= \sup\{\|\Phi(g)\| \mid g \in C_c(R), \|g\|_r \leq 1\} \\ &= \sup\{\|\Phi \circ \Psi(T)\| \mid T \in \text{End}_{A,c}^{00}(X_E), \|T\| \leq 1\} \\ &= \sup\{\|T\| \mid T \in \text{End}_{A,c}^{00}(X_E), \|T\| \leq 1\} \\ &= 1. \end{aligned}$$

Thus, Φ extends to a bounded linear map $\Phi: C^*(R) \rightarrow \text{End}_A^0(X_E)$, which is inverse to Ψ . \square

The description of $\text{End}_A^0(X_E)$ as a groupoid C^* -algebra allows for a concrete description in some instances.

Example 3.2.15. Let $(\mathbb{A}, \Gamma' = \{\gamma_1, \gamma_2\})$ be the tent map iterated function system from Example 2.2.3 and Example 3.2.8, and let X_E be the Kajiwara-Watatani correspondence of Example 3.1.12. It follows from Proposition 3.2.14 that $\text{End}_A^0(X_E)$ is isomorphic to $C^*(R)$, where $R = \{((x, y), (x', y)) \in \text{Gr}(\Gamma') \times \text{Gr}(\Gamma')\}$. Since (\mathbb{A}, Γ') is invertible, Proposition 2.5.8 implies that X_E can be identified with A_σ , where $\sigma: \mathbb{A} \rightarrow \mathbb{A}$ is the tent map. Hence, we can identify $R = \{(x, x') \in \mathbb{A} \mid \sigma(x) = \sigma(x')\}$. It is now straightforward to check that $C^*(R)$ is isomorphic to

$$B := \left\{ f: [0, 1] \rightarrow M_2(\mathbb{C}) \mid f(1) = \begin{pmatrix} \alpha & \alpha \\ \alpha & \alpha \end{pmatrix} \text{ for some } \alpha \in \mathbb{C} \right\}.$$

Since X_E is an $\text{End}_A^0(X_E)$ - A -imprimitivity bimodule B is Morita equivalent to A . Chasing definitions, the isomorphism $\Psi: \text{End}_A^0(X_E) \rightarrow B$ satisfies

$$\Psi(\Theta_{\xi, \eta})(x) = \begin{pmatrix} \xi(\gamma_1(x), x) \overline{\eta(\gamma_1(x), x)} & \xi(\gamma_1(x), x) \overline{\eta(\gamma_2'(x), x)} \\ \xi(\gamma_2'(x), x) \overline{\eta(\gamma_1(x), x)} & \xi(\gamma_2'(x), x) \overline{\eta(\gamma_2'(x), x)} \end{pmatrix}$$

for all $\xi, \eta \in C(\text{Gr}(\Gamma'))$. \triangle

The construction of R can be extended to the path quivers $E^{(n)} = (E^0, E^n, r, s, \lambda^n)$.

Definition 3.2.16. For each $n \in \mathbb{N}_0$ define an equivalence relation

$$R_n := \{(x, y) \in E^n \times E^n \mid s_n(x) = s_n(y)\}.$$

If $(x, y) \in R_n$ we say that x and y are s_n -equivalent.

Let $\mu_n = \{\mu_n^x\}_{x \in R_n^{(0)}}$ denote the Haar system on R_n obtained from λ^n as in (3.9). Then R_n is an amenable topological groupoid with Haar system, and we have an immediate corollary to Proposition 3.2.14.

Corollary 3.2.17. *The groupoid C^* -algebra $C^*(R_n)$ is isomorphic to $\text{End}_A^0(X_E^{\otimes n})$.*

Definition 3.2.18. For each $n \in \mathbb{N}$ define subgroupoids R_n^{reg} and R_n^{sing} of R_n by

$$\begin{aligned} R_n^{reg} &:= \{(x, y) \in E^n \times E^n \mid s(x) = s(y) \in E_{reg}^0\}; \text{ and} \\ R_n^{sing} &:= \{(x, y) \in E^n \times E^n \mid s(x) = s(y) \in E_{sing}^0\}. \end{aligned}$$

By definition of R_n , $s_n^{-1}(E_{reg}^0)$ is an open R_n -invariant subset of $R_n^{(0)} \simeq E^n$, and $R_n^{reg} = R_n|_{s_n^{-1}(E_{reg}^0)}$. Theorem B.2.9 yields a short exact sequence

$$0 \longrightarrow C^*(R_n^{reg}) \xrightarrow{\iota} C^*(R_n) \xrightarrow{q} C^*(R_n^{sing}) \longrightarrow 0, \quad (3.12)$$

with ι induced by the inclusion of $C_c(R_n^{reg})$ into $C_c(R_n)$, and q is induced by the restriction of functions in $C_c(R_n)$ to R_n^{sing} . The Haar system on R_n restricts to Haar systems on R_n^{reg} and R_n^{sing} . We have following corollary to Proposition 3.2.14.

Corollary 3.2.19. *The groupoid C^* -algebra $C^*(R_n^{reg})$ is isomorphic to $\text{End}_A^0(X_E^{\otimes n} \cdot I_{X_E})$.*

Proof. It follows from Lemma 3.1.17 that $X_E^{\otimes n} \cdot I_{X_E}$ is isomorphic to a module obtained by completing $C_c(s_n^{-1}(E_{reg}^0))$. We can then apply the proof of Proposition 3.2.14 mutatis mutandis to $\text{End}_{C_0(E_{reg}^0)}^0(\overline{C_c(s_n^{-1}(E_{reg}^0))})$ and $C^*(R_n^{reg})$. \square

Reflecting on Proposition 3.2.5 our next task is to understand $T(S \otimes \text{id}_{m-n})$ and $(T \otimes \text{id}_{n-m})S$ on the level of groupoids. We define the following product between groupoid algebras.

Proposition 3.2.20. *For each $m \leq n$ let $\phi_m^n: \text{End}_A^0(X_E^{\otimes m}) \rightarrow \text{End}_A(X_E^{\otimes n})$ denote the $*$ -homomorphism $\phi_m^n(T) = T \otimes \text{id}_{n-m}$. For each $n, m \in \mathbb{N}_0$ define a bilinear map $*_{n,m}: C_c(R_n) \times C_c(R_m) \rightarrow C_c(R_{\max\{m,n\}})$ by*

$$(f *_{n,m} g)(x, y) = \begin{cases} \int_{E^m} f(x, zy_{[m+1,n]})g(z, y_{[0,m]}) d\lambda_{r(y_n)}^m(z) & \text{if } n > m \\ (f * g)(x, y) & \text{if } n = m \\ \int_{E^n} f(x_{[0,n]}, z)g(zx_{[n+1,m]}, y) d\lambda_{r(x_m)}^n(z) & \text{if } n < m \end{cases} \quad (3.13)$$

for all $f \in C_c(R_n)$ and $g \in C_c(R_m)$. Then,

$$(f *_{n,m} g) *_{\max\{n,m\},k} h = f *_{n,\max\{m,k\}} (g *_{m,k} h) \quad (3.14)$$

for all $f \in C_c(R_n)$, $g \in C_c(R_m)$, and $h \in C_c(R_k)$. Let $\Phi_k: C^*(R_k) \rightarrow \text{End}_A^0(X_E^{\otimes k})$ be the isomorphism of Corollary 3.2.17. Then,

$$\Phi_{\max\{m,n\}}(f *_{n,m} g) = \phi_n^{\max\{m,n\}}(\Phi_n(f))\phi_m^{\max\{m,n\}}(\Phi_m(g)). \quad (3.15)$$

Moreover, $*_{n,m}$ extends to a bilinear map $*_{n,m}: C^*(R_n) \times C^*(R_m) \rightarrow C^*(R_{\max\{m,n\}})$ satisfying (3.14) and (3.15), and such that $\|a *_{n,m} b\| \leq \|a\|\|b\|$ for all $a \in C^*(R_n)$ and $b \in C^*(R_m)$.

Proof. Since Φ_n is an isomorphism for each $n \in \mathbb{N}$, the associativity condition follows immediately from (3.15). Indeed,

$$\begin{aligned}
\Phi_{\max\{m,n,k\}}((f *_{m,n} g) *_{\max\{m,n\},k} h) \\
&= \phi_{\max\{m,n\}}^{\max\{m,n,k\}}(\Phi_{\max\{m,n\}}(f *_{m,n} g)) \phi_k^{\max\{m,n,k\}}(\Phi_k(h)) \\
&= \phi_{\max\{m,n\}}^{\max\{m,n,k\}}(\phi_n^{\max\{m,n\}}(\Phi_n(f)) \phi_m^{\max\{m,n\}}(\Phi_m(g))) \phi_k^{\max\{m,n,k\}}(\Phi_k(h)) \\
&= \phi_n^{\max\{m,n,k\}}(\Phi_n(f)) \phi_m^{\max\{m,n,k\}}(\Phi_m(g)) \phi_k^{\max\{m,n,k\}}(\Phi_k(h)) \\
&= \Phi_{\max\{m,n,k\}}(f *_{n,\max\{m,k\}} (g *_{m,k} h)).
\end{aligned}$$

The identity (3.15) is obvious in the case where $m = n$. We show it for $n > m$, as the $n < m$ case follows a symmetrically. Let $\xi \in C_c(E^n)$, $\xi_m \in C_c(E^m)$, $\xi_{n-m} \in C_c(E^{n-m})$, and suppose that $\xi(x) = \xi_m(x_{[0,m]})\xi_{n-m}(x_{[m+1,n]})$ for all $x \in E^n$. It follows from Lemma 3.1.8 that such elements are dense in $X_E^{\otimes n}$. Let $f \in C_c(R_n)$ and $g \in C_c(R_m)$. Using Equation (3.11) we compute:

$$\begin{aligned}
&(\Phi_n(f)\phi_m^n(\Phi_m(g)))\xi(x) \\
&= \int_{E^n} f(x, y)(\Phi_m(g) \otimes \text{id}_{n-m})\xi(y) d\lambda_{s(x)}^n(y) \\
&= \int_{E^n} f(x, y)(\Phi_m(g)\xi_m(y_{[0,m]}))\xi_{n-m}(y_{[m+1,n]}) d\lambda_{s(x)}^n(y) \\
&= \int_{E^n} \int_{E^m} f(x, y)g(y_{[0,m]}, z)\xi_m(z) d\lambda_{r(y_{m+1})}^m(z)\xi_{n-m}(y_{[m+1,n]}) d\lambda_{s(x)}^n(y) \\
&= \int_{E^{m-n}} \int_{E^m} \int_{E^m} f(x, y'y'')g(y', z)\xi_m(z) d\lambda_{r(y'')}^m(z)d\lambda_{r(y'')}^m(y') \\
&\quad \times \xi_{n-m}(y'') d\lambda_{s(x)}^{n-m}(y'').
\end{aligned}$$

The functions f, g, ξ_m , and ξ_{n-m} are all compactly supported and the measure $\lambda_{r(y'')}^m$ is σ -finite by assumption. Hence, we can use Fubini's Theorem [Rud87, Theorem 8.8] to see that,

$$\begin{aligned}
&(\Phi_n(f)\phi_m^n(\Phi_m(g)))\xi(x) \\
&= \int_{E^{m-n}} \int_{E^m} \int_{E^m} f(x, y'y'')g(y', z)\xi_m(z) d\lambda_{r(y'')}^m(y')d\lambda_{r(y'')}^m(z)\xi_{n-m}(y'') d\lambda_{s(x)}^{n-m}(y'') \\
&= \int_{E^n} \left(\int_{E^m} f(x, y'w_{[m+1,n]})g(y', w_{[0,m]}) d\lambda_{r(w_{m+1})}^n(y') \right) \xi(w) d\lambda_{s(x)}^n(w) \\
&= \int_{E^n} (f *_{n,m} g)(x, w)\xi(w) d\lambda_{s(x)}^n(w) \\
&= \Phi_n(f *_{n,m} g)\xi(x).
\end{aligned}$$

It now follows that $\Phi_n(f)\phi_m^n(\Phi_m(g)) = \Phi_n(f *_{n,m} g)$ for $n > m$. The final statement follows from Equation (3.15) since $\Phi_{\max\{m,n\}}$ is an isomorphism. \square

3.2.3 The map $T \mapsto T \otimes \text{id}$ groupoidified

Recall from Lemma A.2.4 that the map $T \mapsto T \otimes \text{id}$ from $\text{End}_A(E^{\otimes n})$ to $\text{End}_A(E^{\otimes n+1})$ restricts to an injective $*$ -homomorphism from $\text{End}_A^0(E^{\otimes n} \cdot I_{X_E})$ to $\text{End}_A^0(E^{\otimes n+1})$. From Section 3.2.2 both $\text{End}_A^0(E^{\otimes n} \cdot I_{X_E})$ and $\text{End}_A^0(E^{\otimes n+1})$ admit groupoid models given by $C^*(R_n^{\text{reg}})$ and $C^*(R_{n+1})$, respectively. The aim of this section is to induce the map $T \mapsto T \otimes \text{id}$ from $\text{End}_A^0(E^{\otimes n} I_{X_E})$ to $\text{End}_A^0(E^{\otimes n+1})$ from maps between the groupoids R_n^{reg} and R_{n+1} , giving us groupoid-level description of the left-hand vertical arrow of the diagram (3.7). This is achieved in Proposition 3.2.28.

Recall from Definition C.3.1 that a map is perfect if it is a continuous, proper, surjection. On the level of groupoids, the map $T \rightarrow T \otimes \text{id}$ is not induced by a perfect groupoid homomorphism as in Proposition B.2.7, nor by an open inclusion as in Proposition B.2.8. Instead, it is induced by a combination of the two types of map, together with a normalising factor due to the Haar systems involved.

Pairs of maps between groupoids consisting of a Haar-system-preserving, perfect groupoid homomorphism together with an open inclusion were considered by Austin and Mitra [AM18] who call such pairs *partial morphisms* of groupoids. Partial morphisms are defined later in Definition 3.3.29 in the context of topological graphs. The construction of $T \mapsto T \otimes \text{id}$ in this section does not fall precisely into the partial morphism framework, as the perfect map we construct is not typically Haar-system-preserving. This is a feature of the non-étale setting.

In the process of describing the map $T \mapsto T \otimes \text{id}$, we see precisely where each of the assumptions on the set of regular vertices is used. We also see the markedly different role that branched vertices E_{br}^0 and infinite receivers E_{inf}^0 each play, justifying our distinction. To begin, we introduce the following map which plays a central role in the remainder of this chapter.

Definition 3.2.21. For each $n \in \mathbb{N}_0$ define a continuous map $\rho_n: E^{n+1} \rightarrow E^n$ by,

$$\rho_n(x_1 \cdots x_{n+1}) = \begin{cases} x_1 \cdots x_n & \text{if } n \geq 1; \\ r(x) & \text{if } n = 0. \end{cases} \quad (3.16)$$

For each $n \in \mathbb{N}$, consider the subset

$$F^{n+1} := \{x \in E^{n+1} \mid x_{[0,n]} \in E_{reg}^n\} \subseteq E^{n+1}.$$

As a space F^{n+1} can be realised as the fibre product $F^{n+1} \simeq E_{reg}^n \times_{s_n, r} r^{-1}(E_{reg}^0)$. Since E_{reg}^0 is open in E^0 , F^{n+1} is open in E^{n+1} by Lemma C.1.1. Consider the restriction of ρ_n to F^{n+1} . The fact that E_{reg}^0 contains no sources nor infinite receivers is emphasised in the following lemma.

Lemma 3.2.22. For each $n \in \mathbb{N}_0$, the map $\rho_n: F^{n+1} \rightarrow E_{reg}^n$ is perfect.

Proof. Since E_{reg}^n contains no sources, for each $x \in E_{reg}^n$ there exists $z \in E^1$ such that $s_n(x) = r(z)$. Then ρ_n is surjective since $\rho_n(xz) = x$. For properness fix a compact subset $K \subseteq E_{reg}^n$. Then $s_n(K)$ is compact in E_{reg}^0 . Since E_{reg}^0 contains no infinite receivers, the set $r^{-1}(s_n(K))$ is compact in E^1 . Lemma C.1.1 now implies that $\rho_n^{-1}(K) = K \times_{s_n, r} r^{-1}(s_n(K))$ is compact in F^{n+1} . \square

Recall from Appendix B.4 the notion of a groupoid action. For each $n \in \mathbb{N}_0$ the groupoid R_n acts canonically on its unit space $R_n^{(0)} \simeq E^n$. We extend this action to an action on E^{n+1} by acting on initial segments of a paths.

Lemma 3.2.23. *For each $n \in \mathbb{N}_0$, E^{n+1} is an R_n -space with moment map $\rho_n: E^{n+1} \rightarrow R_n^{(0)}$ and left action $\cdot: R_n \times_{s_{R_n}, \rho_n} E^{n+1} \rightarrow E^{n+1}$ defined for $(x, y) \in R_n$ and $z \in E^{n+1}$ with $y = z_{[0, n]}$ by*

$$(x, y) \cdot z = xz_{n+1}. \quad (3.17)$$

The same formula restricts to an action of R_n^{reg} on F^{n+1} , so that F^{n+1} is an R_n^{reg} -space.

Proof. The formula (3.17) clearly defines a groupoid action. To see that the action is continuous, take a convergent sequence $((x^i, y^i), z^i) \rightarrow ((x, y), z)$ in $R_n^{(0)} \times_{s_{R_n}, \rho_n} E^{n+1}$. Then $x^i \rightarrow x$ in E^n and $z_{n+1}^i \rightarrow z_{n+1}$ in E^1 . Since $s(x^i) = s(y^i) = r(z_{n+1}^i)$ it follows that (x^i, z_{n+1}^i) converges to (x, z_{n+1}) in $E^n \times_{s, r} E^1 \simeq E^{n+1}$. Hence, $x^i z_{n+1}^i \rightarrow xz_{n+1}$.

As F^{n+1} can be identified with $E_{\text{reg}}^n \times_{s_n, r} r^{-1}(E^1)$, the above argument holds upon replacing R_n with R_n^{reg} , and E^{n+1} with F^{n+1} . \square

Since F^{n+1} is an R_n^{reg} -space, we can form the transformation groupoid $R_n^{\text{reg}} \ltimes F^{n+1}$ (see Definition B.4.2). The R_n^{reg} -equivariant map $\rho_n: F^{n+1} \rightarrow E^n \simeq R_n^{(0)}$, lifts to a groupoid homomorphism $\rho_n^\times: R_n^{\text{reg}} \ltimes F^{n+1} \rightarrow R_n^{\text{reg}}$ by Lemma B.4.7. As ρ_n is perfect, it follows from Lemma C.3.20 that ρ_n^\times is also perfect. Since the unit space of $R_n^{\text{reg}} \ltimes F^{n+1}$ can be identified with F^{n+1} , Lemma B.4.4 implies that we can induce a Haar system $\nu_{n+1} = \{\nu_{n+1}^x\}_{x \in F^{n+1}}$ on $R_n^{\text{reg}} \ltimes F^{n+1}$ by setting $\nu_{n+1}^x = \mu_n^{\rho_n(x)}$. By construction ρ_n^\times is necessarily Haar system preserving.

As ρ_n^\times is a Haar system preserving, perfect groupoid homomorphism, Proposition B.2.7 implies that ρ_n^\times induces an injective $*$ -homomorphism $\alpha_n: C^*(R_n^{\text{reg}}) \rightarrow C^*(R_n^{\text{reg}} \ltimes F^{n+1})$ satisfying

$$\alpha_n(f)((x, y), z) = f(x, y) \quad (3.18)$$

for all $((x, y), z) \in R_n^{\text{reg}} \ltimes F^{n+1}$ and $f \in C_c(R_n^{\text{reg}})$. The map α_n forms part of our description of $T \mapsto T \otimes \text{id}$.

We divert our attention for the moment and instead consider, for each $n \in \mathbb{N}_0$, the subgroupoid of R_n given by,

$$S_n := \{(x, y) \in R_n \mid x, y \in F^n\} = \{(x, y) \in R_n \mid x_n = y_n \in r^{-1}(E_{\text{reg}}^0)\}. \quad (3.19)$$

Lemma 3.2.24. *For each $n \in \mathbb{N}_0$, S_n is an open subgroupoid of R_n .*

Proof. Let $(x, y) \in S_n$. Since $r(x_n) \in E_{\text{reg}}^0 \subseteq E^0 \setminus E_{br}^0$ there is an open neighbourhood $V \subseteq E_{\text{reg}}^0$ of $r(x_n)$ such that $s|_{r^{-1}(V)}$ is a local homeomorphism. In particular, we can find an open neighbourhood $U \subseteq r^{-1}(V)$ of x_n such that $s|_U$ is injective. Then $W := E^{n-1} \times_{s, r} U$ is an open neighbourhood of both x and y , so that $(W \times W) \cap R_n$ is an open neighbourhood of (x, y) . Now, if $(x', y') \in (W \times W) \cap R_n$ then $x'_n, y'_n \in U$ and $s(x'_n) = s(y'_n)$. Injectivity of $s|_U$ then implies that $x'_n = y'_n$, so $(x', y') \in S_n$. Hence, $(W \times W) \cap R_n$ is an open neighbourhood of (x, y) contained in S_n . \square

Lemma 3.2.24 highlights the importance of excluding branched vertices from the definition of E_{reg}^0 . If E_{br}^0 were non-empty, and we instead considered the subgroupoid of R_n consisting of pairs $(x, y) \in R_n$ such that $x_n = y_n \in r^{-1}(E_{\text{reg}}^0)$, then the analogue of Lemma 3.2.24 would fail.

Equip S_n with the Haar system $\mu_n = \{\mu_n^x\}_{x \in F^n}$ it inherits as an open subgroupoid of R_n . It follows from Proposition B.2.8 that the inclusion of S_n into R_n induces a $*$ -homomorphism $\beta_n: C^*(S_n) \rightarrow C^*(R_n)$. The map β_n also forms part of our description of $T \mapsto T \otimes \text{id}$.

Lemma 3.2.25. *For each $n \in \mathbb{N}_0$ there is a isomorphism $\sigma_{n+1}: S_{n+1} \rightarrow R_n^{\text{reg}} \ltimes F^{n+1}$ of topological groupoids given by*

$$\sigma_{n+1}(x, y) = ((x_{[0,n]}, y_{[0,n]}), y)$$

for all $(x, y) \in S_{n+1}$.

Proof. To see that σ_{n+1} is a groupoid homomorphism let $(x, y), (y, z) \in S_{n+1}$. Since $y = (y_{[0,n]}, z_{[0,n]}) \cdot z$ it follows that

$$\begin{aligned} \sigma_{n+1}(x, y)\sigma_{n+1}(y, z) &= ((x_{[0,n]}, y_{[0,n]}), y)((y_{[0,n]}, z_{[0,n]}), z) \\ &= ((x_{[0,n]}, z_{[0,n]}), z) \\ &= \sigma_{n+1}((x, y)(y, z)). \end{aligned}$$

The inverse of σ_{n+1} is given by $\sigma_{n+1}^{-1}((x, y), z) = (xz_{n+1}, yz_{n+1})$. Similar sequence arguments to those in the proof of Lemma 3.2.23 show that σ_{n+1} and σ_{n+1}^{-1} are both continuous. \square

Although S_{n+1} and $R_n^{\text{reg}} \ltimes F^{n+1}$ are isomorphic as topological groupoids, there is a subtlety when we consider them with their associated Haar systems. Let $\sigma_{n+1}^*: C_c(R_n^{\text{reg}} \ltimes F^{n+1}) \rightarrow C_c(S_{n+1})$ be the $*$ -homomorphism dual to σ_{n+1} . That is,

$$\sigma_{n+1}^*(f)(x, y) = ((x_{[0,n]}, y_{[0,n]}), y).$$

The Haar system on $R_n^{\text{reg}} \ltimes F^{n+1}$ was induced from the Haar system on R_n^{reg} , while the Haar system on S_{n+1} was inherited from R_{n+1} . With some careful definition chasing, for each $f \in C_c(R_n^{\text{reg}} \ltimes F^{n+1})$, we compute:

$$\begin{aligned} \mu_{n+1}^x(\sigma_{n+1}^*(f)) &= \int_{S_{n+1}^x} (f \circ \sigma_{n+1})(x, y) d\mu_{n+1}^x(x, y) \\ &= \int_{E^{n+1}} \chi_{S_{n+1}}(x, y) (f \circ \sigma_{n+1})(x, y) d\lambda_{s_n(x)}^{n+1}(y) \\ &= \int_{E^1} \int_{E^1} \cdots \int_{E^1} \chi_{S_{n+1}}(x, y_1 \cdots y_{n+1}) (f \circ \sigma_{n+1})(x_{[0,n]}x_{n+1}, y_1 \cdots y_n y_{n+1}) \\ &\quad d\lambda_{r(y_2)}(y_1) \cdots d\lambda_{r_{x_{n+1}}}(y_n) d\lambda_{s_n(x)}(y_{n+1}) \\ &= \int_{E^1} \delta_{x_{n+1}, y_{n+1}} \left(\int_{R_n^{\text{reg}}} f((x_{[0,n]}, y'), y' y_{n+1}) d\mu_n^{x_{[0,n]}}(y') \right) d\lambda_{s_n(x)}(y_{n+1}) \\ &= \int_{E^1} \delta_{x_{n+1}, y_{n+1}} \nu_{n+1}^{x_{[0,n]} y_{n+1}}(f) d\lambda_{s_n(x)}(y_{n+1}) \\ &= \lambda_{s(x_{n+1})}(\{x_{n+1}\}) \nu_{n+1}^x(f). \end{aligned} \tag{3.20}$$

The factor of $\lambda_{s(x_{n+1})}(\{x_{n+1}\})$ means that σ_{n+1} is not Haar system preserving. Consequently, the map $\sigma_{n+1}^*: C_c(R_n^{reg} \ltimes F^{n+1}) \rightarrow C_c(S_{n+1})$ does not immediately induce a $*$ -homomorphism on the level of C^* -algebras. Fortunately though, we can normalise σ_{n+1}^* . This normalisation is heavily dependent on the fact that we have excluded branched vertices from the definition of E_{reg}^0 . In order to normalise we require the following lemma.

Lemma 3.2.26. *The map $x \mapsto \lambda_{s(x)}(\{x\})$ is continuous and strictly positive on $E_{\acute{e}t}^1$.*

Proof. Fix $x \in E_{\acute{e}t}^1$. Then there exists an open neighbourhood U of x such that $s|_U$ is injective. Since U is itself second-countable, locally compact, and Hausdorff we can choose a precompact neighbourhood W of x such that $W \subseteq \bar{W} \subseteq U$. Urysohn's Lemma [Rud87, Theorem 2.12] gives a function $h \in C_c(U) \subseteq C_c(E_{\acute{e}t}^1)$ such that $h|_{\bar{W}} = 1$. Since $s|_U$ is injective, $\text{supp}(h) \cap s^{-1}(v)$ consists of a single point for all $v \in s(U)$. In particular, for all $y \in U$ we have $\lambda_{s(y)}(\{y\}) = \int h d\lambda_{s(y)}$.

Now suppose that $x_n \rightarrow x$ in E^1 . Then there exists $N \in \mathbb{N}$ such that for $n \geq N$ we have $x_n \in U$. Then, for $n \geq N$ we have

$$|\lambda_{s(x)}(\{x\}) - \lambda_{s(x_n)}(\{x_n\})| = \left| \int h d\lambda_{s(x)} - \int h d\lambda_{s(x_n)} \right|. \quad (3.21)$$

Since $v \mapsto \int h d\lambda_v$ is continuous, the right-hand side of (3.21) converges to 0 as $n \rightarrow \infty$. Thus, $x \mapsto \lambda_{s(x)}(\{x\})$ is continuous.

For strict positivity note that since $\lambda_{s(x)}$ is a positive Radon measure with support $s^{-1}(s(x))$ and $U \cap s^{-1}(s(x)) = \{x\}$ we have $\lambda_{s(x)}(\{x\}) > 0$. \square

Proposition 3.2.27. *For each $n \in \mathbb{N}_0$ there is a $*$ -isomorphism $\sigma_{n+1}^\dagger: C_c(R_n^{reg} \ltimes F^{n+1}) \rightarrow C_c(S_{n+1})$ of convolution algebras satisfying*

$$\sigma_{n+1}^\dagger(f)(x, y) = \frac{(f \circ \sigma_{n+1})(x, y)}{\lambda_{s(x_{n+1})}(\{x_{n+1}\})} \quad (3.22)$$

for all $f \in C_c(R_n^{reg} \ltimes F^{n+1})$, and $(x, y) \in S_{n+1}$. Moreover, σ_{n+1}^\dagger extends to an isomorphism of C^* -algebras, $\sigma_{n+1}^\dagger: C^*(R_n^{reg} \ltimes F^{n+1}) \rightarrow C^*(S_{n+1})$.

Proof. Since $(x, y) \in S_{n+1}$ implies $x_{n+1} = y_{n+1} \in r^{-1}(E_{\acute{e}t}^0) \subseteq E_{\acute{e}t}^1$, it follows from Lemma 3.2.26 that the map $(x, y) \mapsto \lambda_{s(x_{n+1})}(\{x_{n+1}\})$ on S_{n+1} is continuous and strictly positive. Since σ_{n+1} is a topological groupoid isomorphism, $\sigma_{n+1}^\dagger: C_c(R_n^{reg} \ltimes F^{n+1}) \rightarrow C_c(S_{n+1})$ is a $*$ -preserving vector-space isomorphism.

For multiplicativity we perform a calculation similar to (3.20) and compute, for $f, g \in C_c(R_n^{reg} \ltimes F^{n+1})$ and $(x, y) \in S_{n+1}$:

$$\begin{aligned} & (\sigma_{n+1}^\dagger(f) * \sigma_{n+1}^\dagger(g))(x, y) \\ &= \frac{1}{\lambda_{s(x_{n+1})}(\{x_{n+1}\})^2} \int_{S_{n+1}^x} (f \circ \sigma_{n+1})(x, z) (g \circ \sigma_{n+1})(z, y) d\mu_{n+1}^x(x, z) \\ &= \frac{1}{\lambda_{s(x_{n+1})}(\{x_{n+1}\})} \int_{(R_n^{reg})^{x_{[1,n]}}} f((x_{[1,n]}, z'), z'x_{n+1}) g((z', y_{[1,n]}), y) d\mu_n^x(x_{[1,n]}, z') \\ &= \frac{1}{\lambda_{s(x_{n+1})}(\{x_{n+1}\})} (f * g)(\sigma_{n+1}(x, y)) \\ &= \sigma_{n+1}^\dagger(f * g)(x, y). \end{aligned}$$

The computation (3.20) shows that $\mu_{n+1}^x(\sigma_{n+1}^\dagger(f)) = \nu_{n+1}^x(f)$ for all $f \in C_c(R_n^{reg} \ltimes F^{n+1})$ and $x \in F^{n+1}$. Consequently, $\|\sigma_{n+1}^\dagger(f)\|_r = \|f\|_r$ for all $f \in C_c(R_n^{reg} \ltimes F^{n+1})$. The final statement now follows. \square

To summarise, there is a $*$ -homomorphism $\alpha_n: C^*(R_n^{reg}) \rightarrow C^*(R_n^{reg} \ltimes F^{n+1})$, an isomorphism $\sigma_{n+1}^\dagger: C^*(R_n^{reg} \ltimes F^{n+1}) \rightarrow C^*(S_{n+1})$, and a $*$ -homomorphism $\beta_{n+1}: C^*(S_{n+1}) \rightarrow C^*(R_{n+1})$. Denote the composition $\beta_{n+1} \circ \sigma_{n+1}^\dagger \circ \alpha_n$ by $\tau_n: C^*(R_n^{reg}) \rightarrow C^*(R_{n+1})$. Then for $f \in C_c(R_n^{reg})$,

$$\tau_n(f)(x, y) = \delta_{x_{n+1}, y_{n+1}} \frac{f(x_{[0, n]}, y_{[0, n]})}{\lambda_{s(x_{n+1})}(\{x_{n+1}\})}. \quad (3.23)$$

As desired, the $*$ -homomorphism τ_n describes the map $T \mapsto T \otimes \text{id}$ from $\text{End}_A^0(X_E^{\otimes n} \cdot I_{X_E})$ to $\text{End}_A^0(X_E^{\otimes n+1})$ on the level of groupoid C^* -algebras.

Proposition 3.2.28. *For each $n \in \mathbb{N}_0$ let Φ_n and Φ_{n+1} be the isomorphisms of Corollary 3.2.19 and Corollary 3.2.17 respectively. Then for each $n \in \mathbb{N}_0$, the diagram*

$$\begin{array}{ccc} C^*(R_n^{reg}) & \xrightarrow{\Phi_n} & \text{End}_A^0(X_E^{\otimes n} \cdot I_{X_E}) \\ \downarrow \tau_n & & \downarrow T \mapsto T \otimes 1 \\ C^*(R_{n+1}) & \xrightarrow{\Phi_{n+1}} & \text{End}_A^0(X_E^{\otimes n+1}) \end{array}$$

commutes.

Proof. Recall from Lemma 3.1.8 that $X_E^{\otimes n}$ is isomorphic to $X_{E^{(n)}}$ as C^* -correspondences over $A = C_0(E^0)$. Let $Z = \{x \mapsto \xi(x_{[1, n]})\eta(x_{n+1}) \mid \xi \in C_c(E^n), \eta \in C_c(E^1)\}$. It follows from Lemma 3.1.8 that $\overline{\text{span}}Z = X_{E^{(n+1)}}$. Take $f \in C_c(R_n^{reg})$ and suppose that $\zeta \in \Sigma$ is of the form $\zeta(x) = \xi(x_{[1, n]})\eta(x_{n+1})$ for $\xi \in C_c(E^n)$ and $\eta \in C_c(E^1)$. Then,

$$\begin{aligned} ((\Phi_{n+1} \circ \tau_n)(f)\zeta)(x) &= \int_{E^{n+1}} \delta_{x_{n+1}, y_{n+1}} \frac{f(x_{[1, n]}, y_{[1, n]})}{\lambda_{s(x_{n+1})}(x_{n+1})} \zeta(y) d\lambda_{s_{n+1}(x)}^{n+1}(y) \\ &= \int_{E^{n+1}} \delta_{x_{n+1}, y_{n+1}} \frac{f(x_{[1, n]}, y_{[1, n]})}{\lambda_{s(x_{n+1})}(x_{n+1})} \xi(y_{[1, n]})\eta(y_{n+1}) d\lambda_{s_{n+1}(x)}^{n+1}(y) \\ &= \int_{E^1} \frac{\eta(x_{n+1})}{\lambda_{s(x_{n+1})}(x_{n+1})} \left(\int_{E^n} f(x_{[1, n]}, y') \xi(y') d\lambda_{r(x_{n+1})}^n(y') \right) d\lambda_{s(x_{n+1})}(y'') \\ &= \int_{E^1} \frac{\eta(x_{n+1})}{\lambda_{s(x_{n+1})}(x_{n+1})} \Phi_n(f) \xi(x_{[1, n]}) d\lambda_{s(x_{n+1})}(y'') \\ &= (\Phi_n(f)\xi)(x_{[1, n]})\eta(x_{n+1}). \end{aligned}$$

Since $\overline{\text{span}}Z = X_{E^{(n+1)}}$, upon identifying $X_{E^{(n)}} \otimes_A X_E$ with $X_{E^{(n+1)}}$ via the map Γ of Lemma 3.1.8, we see that $(\Phi_{n+1} \circ \tau_n)(f) = \Phi_n(f) \otimes \text{id}$. Continuity gives the extension to $C^*(R_n^{reg})$. \square

3.3 | Reconstructing the core

Now that we have described a $*$ -homomorphism $\tau_n: C^*(R_n^{reg}) \rightarrow C^*(R_{n+1})$ built from groupoid implementing the map $T \mapsto T \otimes 1$ from the diagram (3.7), we move on to the

harder task of determining groupoid models for $B_{[0,n]}$ and $C_{[0,n]}$. Simultaneously, we need to determine how the inclusions $\iota_n: B_{[0,n]} \rightarrow B_{[0,n+1]}$ and $\iota_n: C_{[0,n]} \rightarrow C_{[0,n+1]}$ are realised at the groupoid level.

Let us consider, for the moment, the case when $n = 0$. In this case, we require a groupoid G_1 such that $C^*(G_1)$ is isomorphic $B_{[0,1]}$. The split exact sequence (3.8) implies that there is an inclusion of $C^*(R_1) \cong \text{End}_A^0(X_E)$ into $C^*(G_1)$ as an ideal, and that the inclusion $\iota_1: C_0(E^0) \rightarrow C^*(G_1)$ must split the quotient map. Typically, $C^*(G_1)$ is not a direct sum of $C^*(R_1)$ and $C_0(E^0)$.

Inspired by the role played by $\rho_n: E^{n+1} \rightarrow E^n$ in the construction of τ_n , we use ρ_1 in order to glue R_1 to $E^0 \cong R_0$ in an appropriate way, resulting in the desired groupoid G_1 . As a first hurdle, unlike in Lemma 3.2.22, the map $\rho_n: E^{n+1} \rightarrow E^n$ is typically not perfect: sources prevent surjectivity, and infinite receivers prevent properness. Consequently, the dual map $\rho_n^*: C(E^n) \rightarrow C(E^{n+1})$ does not take compactly supported functions to compactly supported functions. Moreover, when E is a topological graph, the unit spaces of each G_n should assemble—via a limit construction—into the unit space $E^{\leq \infty}$ of \mathcal{TR}_E . In the Cuntz-Pimsner case $E^{\leq \infty}$ should be substituted with ∂E . In the next section we overcome the fact ρ_n is not perfect using a topological construction called perfection.

In graph C^* -algebra literature, the presence of sources is often overcome by the process of “adding tails” to the graph. This process is described for topological quivers in [MT05b, §4]. Adding tails to a topological quiver modifies the quiver, and the resulting Cuntz-Pimsner algebra is Morita equivalent to the algebra with which we started. Since we are aiming to construct a precise isomorphism between \mathcal{O}_{X_E} and a groupoid C^* -algebra, we avoid this approach.

3.3.1 The boundary path space: perfected

As mentioned in the introduction to this section, the map $\rho_n: E^{n+1} \rightarrow E^n$ is not perfect and therefore does not induce an injective $*$ -homomorphism from $C_0(E^n)$ to $C_0(E^{n+1})$. We extend ρ_n to a perfect map via the notion of perfections. Perfections are examined in detail in Appendix C.3.

Briefly, a perfection of a continuous map $p: X \rightarrow Y$ consists of a new space Z , containing X , together with an extension of p from Z to Y which is a perfect map. The space Z is typically constructed by gluing elements of Y onto X in a way that is compatible with p . This is akin to compactifying a space, but the resulting space Z is not necessarily compact (see Example C.3.7), and the original space may not be dense in the new space (it is if p is surjective). Perfections are an extension of the idea of compactifications of mappings introduced by Whyburn [Why53; Why66], which are constructed in the case where p is already surjective. In the C^* -algebraic context, fibrewise compactifications were recently considered by Anantharaman-Delaroche in the unpublished work [AD14].

As a by-product of turning ρ_n into a perfect map, we uncover a new inverse limit construction of both the path space $E^{\leq \infty}$ and boundary path space ∂E , introduced in Section 3.1.4. As far as the author is aware, a natural inverse limit construction of ∂E has not been written down. This construction is the main focus of this section.

Historically, the boundary path space was first considered by Paterson [Pat02] in the context of directed graphs. Paterson constructed the boundary path space of a directed graph by considering semicharacters on idempotents in the inverse semigroup associated to \mathcal{G}_E . The boundary path space was later studied by Webster [Web14] who instead realised it as a subspace of the infinite path space of a Drinen-Tomforde desingularisation

[DT05] of E . Yeend [Yee07] extended the definition to work for topological higher-rank graphs. Recently, de Castro [dCa18] investigated boundary path spaces for topological graphs using pointless topology.

A major advantage of our construction is that it is built iteratively from the bottom-up using the finite paths of E , and does not rely on any knowledge of an ambient C^* -algebra or groupoid. Although we have limited ourselves to studying the boundary path space of a topological quiver, the process we outline in the sequel can be applied to turn any inverse system (indexed by \mathbb{N}) of locally compact Hausdorff spaces with continuous maps into an inverse system of locally compact Hausdorff spaces with perfect maps. Moreover, the inverse limit of the original system sits as a subspace of the inverse limit of the new system. The full scope of this construction remains to be explored.

To begin, observe that the maps $\rho_n: E^{n+1} \rightarrow E^n$ of Definition 3.2.21 may be assembled into an inverse system of topological spaces (E^n, ρ_n) . As a set, the inverse limit $\varprojlim (E^n, \rho_n)$ can be realised as the collection of *infinite paths* in E ,

$$E^\infty := \left\{ x_1 x_2 \cdots \in \prod_{\mathbb{N}} E^1 \mid s(x_i) = r(x_{i+1}) \text{ for all } i \in \mathbb{N} \right\}.$$

If $E_{rsing}^0 \neq \emptyset$, the bonding maps ρ_n are neither surjective (due to sources) nor proper (due to infinite receivers). Consequently, the initial topology on E^∞ is typically not locally compact.

Example 3.3.1. Consider the directed graph $E = (E^0, E^1, r, s)$ where $E^0 = \{v\}$ and $E^1 = \{e_i \mid i \in \mathbb{N}\}$. Suppose that $r(e_i) = s(e_i) = v$ for all $i \in \mathbb{N}$. The reader familiar with graph C^* -algebras may realise that the associated graph C^* -algebra $C^*(E)$ is isomorphic to the Cuntz algebra \mathcal{O}_∞ . In this case, $E^n = \prod_{i=1}^n E^1$ is discrete. Consider the inverse limit $E^\infty = \varprojlim (E^n, \rho_n)$ equipped with the initial topology induced by the universal maps $\rho_{\infty, n}: E^\infty \rightarrow E^n$. A basis of clopen sets for the topology on E^∞ is given by $\{\rho_{\infty, n}^{-1}(x) \mid x \in E^n, n \in \mathbb{N}\}$. Then E^∞ is Hausdorff, but we claim that it is not locally compact.

We begin by showing that $\rho_{\infty, n}^{-1}(x)$ is not compact for any $x \in E^n$ and $n \in \mathbb{N}$. Indeed, we can write $\rho_{\infty, n}^{-1}(x) = \bigsqcup_{i \in \mathbb{N}} \rho_{\infty, n}^{-1}(x e_i)$, so that the open cover $\{\rho_{\infty, n}^{-1}(x e_i)\}_{i \in \mathbb{N}}$ of $\rho_{\infty, n}^{-1}(x)$ admits no finite subcover.

Suppose for contradiction that E^∞ is locally compact. Fix $x \in E^n$. By local compactness, there is a precompact open set W such that $\rho_{\infty, n}^{-1}(x) \subseteq W$. Since $\rho_{\infty, n}^{-1}(x)$ is clopen, it would be a closed subset of \overline{W} which is not compact. Since E^∞ is Hausdorff this is impossible. Hence, E^∞ is not locally compact. \blacktriangle

In general, inverse limits behave in the following way with respect to proper and perfect maps.

Proposition 3.3.2 ([Eng89, Theorem 3.7.13]). *Let \mathcal{A} be a directed set and suppose that $(\{X_\alpha\}_{\alpha \in \mathcal{A}}, \{\rho_{\alpha, \beta}\}_{\alpha, \beta \in \mathcal{A}})$ is an inverse system of locally compact spaces X_α with proper maps $\rho_{\alpha, \beta}: X_\alpha \rightarrow X_\beta$. Then the inverse limit space $\varprojlim (X_\alpha, \rho_{\alpha, \beta})$ is a locally compact space in the initial topology defined by the natural projections $\rho_{\infty, \alpha}: \varprojlim (X_\alpha, \rho_{\alpha, \beta}) \rightarrow X_\alpha$, and the maps $\rho_{\infty, \alpha}$ are proper.*

Corollary 3.3.3. *Let \mathcal{A} be a countable directed set and let $(\{X_\alpha\}_{\alpha \in \mathcal{A}}, \{\rho_{\alpha, \beta}\}_{\alpha, \beta \in \mathcal{A}})$ be an inverse system consisting of second-countable locally compact Hausdorff spaces X_α and perfect maps $\rho_{\alpha, \beta}: X_\alpha \rightarrow X_\beta$. Then $\varprojlim (X_\alpha, \rho_{\alpha, \beta})$ is a second-countable locally compact Hausdorff space, and the maps $\rho_{\infty, \alpha}: \varprojlim (X_\alpha, \rho_{\alpha, \beta}) \rightarrow X_\alpha$ are perfect.*

Proof. The properness and Hausdorffness of $\varprojlim (X_\alpha, \rho_{\alpha,\beta})$ follows directly from Proposition 3.3.2. Second-countability is immediate from the definition of the initial topology, since the space X_α are second-countable and \mathcal{A} is countable. Surjectivity of the maps $\{\rho_{\infty,\alpha}\}_{\alpha \in \mathcal{A}}$ follows from surjectivity of the maps $\{\rho_{\alpha,\beta}\}_{\alpha,\beta \in \mathcal{A}}$. \square

Remark 3.3.4. If $(\{X_\alpha\}_{\alpha \in \mathcal{A}}, \{\rho_{\alpha,\beta}\}_{\alpha,\beta \in \mathcal{A}})$ is an inverse system satisfying the hypotheses of Corollary 3.3.3, then it induces a directed system $(\{C_0(X_\alpha)\}_{\alpha \in \mathcal{A}}, \{\rho_{\alpha,\beta}^*\}_{\alpha,\beta \in \mathcal{A}})$ of C^* -algebras, where $\rho_{\alpha,\beta}^*: C_0(X_\beta) \rightarrow C_0(X_\alpha)$ is the injective $*$ -homomorphism given $\rho_{\alpha,\beta}^*(f)(x) = f(\rho_{\alpha,\beta}(x))$. Then $C_0(\varprojlim (X_\alpha, \rho_{\alpha,\beta})) \cong \varinjlim (C_0(X_\alpha), \rho_{\alpha,\beta}^*)$ by universality of limits.

The aim for the remainder of this section is to iteratively use perfections of the maps $\rho_n: E^{n+1} \rightarrow E^n$ so that we may apply Corollary 3.3.3. As it turns out, when E is a topological graph, the two types of perfection described in Appendix C.3—the *unified space* and the *minimal perfection*—yield the path space $E^{\leq \infty}$, and boundary path space ∂E of E , respectively. If E is a topological quiver, the only difference is that the boundary path space only includes finite paths whose source is r -singular. The branched vertices play no role.

3.3.1.1 The path space $E^{\leq \infty}$

We begin by constructing the *path space*

$$E^{\leq \infty} := E^\infty \sqcup \bigsqcup_{k=0} E^k$$

using perfections. As a topological space, $E^{\leq \infty}$ is typically not equipped with the disjoint union topology (cf. [Web14; Yee07; dCa18]). The topology on $E^{\leq \infty}$ is recovered (and perhaps explained) with our construction.

To obtain the path space we make repeated application of the unified space construction (Definition C.3.2). To begin, consider the map $r: E^1 \rightarrow E^0$. Recall from Appendix C.3.1 that the unified space $(\widehat{E^1})_r$ is $E^1 \sqcup E^0$ as a set. Define $\tilde{r}: E^1 \sqcup E^0 \rightarrow E^0$ by

$$\tilde{r}(x) = \begin{cases} r(x) & \text{if } x \in E^1; \text{ and} \\ x & \text{if } x \in E^0. \end{cases}$$

and equip $E^1 \sqcup E^0$ with the topology generated by the basic open sets

$$\{U \mid U \text{ open in } E^1\} \cup \{\tilde{r}^{-1}(V) \cap (E^{\leq 1} \setminus K) \mid V \text{ is open in } E^0 \text{ and } K \subseteq E^1 \text{ is compact}\}.$$

Theorem C.3.6 implies that \tilde{r} is a perfect map. For each $n \in \mathbb{N}_0$ denote the set $\bigsqcup_{k=0}^n E^k$ by $E^{\leq n}$. We identify $(\widehat{E^1})_r$ with $E^{\leq 1}$. Theorem C.3.6 implies that E^1 is open in $E^{\leq 1}$ and E^0 is closed in $E^{\leq 1}$. We often use this fact without reference.

Remark 3.3.5. If $E_{rsing}^0 = \emptyset$, the topology on $E^{\leq 1}$ is just the disjoint union topology. To see why, fix a precompact open subset V of E^0 . Then $r^{-1}(\overline{V})$ is compact in E^1 . Consequently, $\tilde{r}^{-1}(V) \cap E^{\leq 1} \setminus r^{-1}(\overline{V}) = V$ is open in the unified space and contained in $E^0 \subseteq E^1 \sqcup E^0$. Since precompact open sets form a basis for the topology on E^0 , $E^{\leq 1}$ has the disjoint union topology.

Now consider $\rho_1: E^2 \rightarrow E^1$. Composing ρ_1 with the open inclusion $E^1 \hookrightarrow E^{\leq 1}$ yields a continuous map that we also denote $\rho_1: E^2 \rightarrow E^{\leq 1}$. We then construct the unified space $(E^2)_{\rho_1}$ —which we identify with $E^2 \sqcup E^{\leq 1} = E^{\leq 2}$ —and the associated perfect extension $\tilde{\rho}_1: E^{\leq 2} \rightarrow E^{\leq 1}$ of ρ_1 . Proceeding inductively we induce topologies on $E^{\leq n}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}_0$ there is a perfect maps $\tilde{\rho}_n: E^{\leq n+1} \rightarrow E^{\leq n}$ given by,

$$\tilde{\rho}_n(x) = \begin{cases} \rho_n(x) & \text{if } x \in E^{n+1}, \\ x & \text{if } x \in E^{\leq n}. \end{cases}$$

Notation 3.3.6. For each $k, n \in \mathbb{N}_0$ with $k \geq n$ define $\tilde{\rho}_{k,n}: E^{\leq k} \rightarrow E^{\leq n}$ by

$$\tilde{\rho}_{k,n} := \tilde{\rho}_n \circ \cdots \circ \tilde{\rho}_{k-1}.$$

Let $\tilde{\rho}_{n,n} = \text{id}_{E^{\leq n}}$.

Corollary 3.3.3 implies the following.

Proposition 3.3.7. *The collection $(E^{\leq n}, \tilde{\rho}_n)_{n \in \mathbb{N}_0}$ forms an inverse system in the category of second-countable, locally compact, Hausdorff spaces with perfect maps. In particular, $\varprojlim (E^{\leq n}, \tilde{\rho}_n)$ is a second-countable, locally compact, Hausdorff space when equipped with the initial topology given by the universal maps $\tilde{\rho}_{\infty,k}: \varprojlim (E^{\leq n}, \tilde{\rho}_n) \rightarrow E^{\leq k}$, and for all $k \in \mathbb{N}_0$, the map $\tilde{\rho}_{\infty,k}$ is perfect.*

As a set, it is straightforward to verify that $\varprojlim (E^{\leq n}, \tilde{\rho}_n)$ can be identified with $E^{\leq \infty}$. Under this identification the universal maps $\tilde{\rho}_{\infty,k}: E^{\leq \infty} \rightarrow E^{\leq k}$ are given by,

$$\tilde{\rho}_{\infty,k}(x) = \begin{cases} x_{[0,k]} & \text{if } x \in E^\infty \sqcup \bigsqcup_{n=k+1}^\infty E^n \\ x & \text{if } \bigsqcup_{n=0}^k E^n. \end{cases}$$

for each $k \in \mathbb{N}_0$. Considering E^k as an open subset of $E^{\leq k}$, note that

$$E^{\leq \infty} \setminus E^{\leq k-1} = \tilde{\rho}_{\infty,k}^{-1}(E^k) \quad (3.24)$$

is open in $E^{\leq \infty}$. We extend the notion of path spaces to topological quivers.

Definition 3.3.8. Let E be a topological quiver. Define the *path space* of E to be the inverse limit space $E^{\leq \infty} := \varprojlim (E^{\leq n}, \tilde{\rho}_n)$ with the initial topology induced by the universal maps $\tilde{\rho}_{\infty,k}: E^{\leq \infty} \rightarrow E^{\leq k}$.

Fix $k \in \mathbb{N}$. We now describe a basis for the topology on $E^{\leq k}$. Suppose that $0 \leq c \leq k$ and that for all $0 \leq m \leq c$, U_m is an open subset of E^m . Define

$$\begin{aligned} \mathcal{Z}_k(U_0, U_1, \dots, U_c) &:= \bigcap_{m=0}^c \tilde{\rho}_{k,m}^{-1}(U_m) \\ &= \{x \in E^{\leq k} \mid x_{[0,m]} \in U_m \text{ for all } 0 \leq m \leq c\} \end{aligned}$$

Similarly, suppose $1 \leq d \leq k$ and for all $1 \leq n \leq d$, K_n is a compact subset of E^n . Define

$$\begin{aligned}\mathcal{Z}_k(K_1, K_2, \dots, K_d) &:= \bigcap_{n=1}^d \tilde{\rho}_{k,n}^{-1}(K_n) \\ &= \{x \in E^{\leq k} \mid x_{[1,n]} \in K_n \text{ for all } 1 \leq n \leq d\}.\end{aligned}$$

Whether we are dealing with compact sets or open sets should be clear from context.

Proposition 3.3.9. *The collection*

$$\begin{aligned}\mathcal{B}_k &:= \{\mathcal{Z}_k(U_0, \dots, U_c) \cap (E^{\leq k} \setminus \mathcal{Z}_k(K_1, \dots, K_d)) \mid 0 \leq c \leq k, 1 \leq d \leq k, U_d \subseteq E^d \text{ open,} \\ &\quad \text{and } K_c \subseteq E^c \text{ compact}\}\end{aligned}$$

is a basis for the unified space topology on $E^{\leq k}$.

Proof. By definition of the unified space topology this is true for $k = 1$. Suppose for induction that \mathcal{B}_k is a basis for the unified space $E^{\leq k}$. By virtue of being defined via preimages, the intersection of two elements of \mathcal{B}_k gives another element of \mathcal{B}_k . Hence, \mathcal{B}_k is a basis for some topology on $E^{\leq k}$. Since each $\tilde{\rho}_l$ is continuous, the sets making up \mathcal{B}_k are open in the unified space topology.

Recall that a basis for the unified space topology on $E^{\leq k+1}$ is given by

$$\begin{aligned}\mathcal{B} &= \{\tilde{\rho}_k^{-1}(V) \cap (E^{\leq k+1} \setminus K) \mid V \subseteq E^{\leq k} \text{ open, and } K \subseteq E^{k+1} \text{ compact}\} \\ &\quad \cup \{U \mid U \text{ open in } E^{k+1}\}\end{aligned}$$

If U is open in E^{k+1} , then

$$U = \mathcal{Z}_{k+1}(E^0, E^1, \dots, E^k, U) \cap (E^{\leq k+1} \setminus \mathcal{Z}_{k+1}(\emptyset, \dots, \emptyset)) \in \mathcal{B}_k.$$

Now suppose that V is open in $E^{\leq k}$ and K is compact in E^{k+1} . By induction there exist U_m open in E^m and K_n compact in E^k such that

$$\mathcal{Z}_k(U_0, \dots, U_c) \cap (E^{\leq k} \setminus \mathcal{Z}_k(K_1, \dots, K_d)) \subseteq V.$$

Then

$$\begin{aligned}\tilde{\rho}_k^{-1}(\mathcal{Z}_k(U_0, \dots, U_c) \cap (E^{\leq k} \setminus \mathcal{Z}_k(K_1, \dots, K_d))) \\ = \mathcal{Z}_{k+1}(U_0, \dots, U_c) \cap (E^{\leq k+1} \setminus \mathcal{Z}_{k+1}(K_1, \dots, K_d))\end{aligned}$$

is a subset of $\tilde{\rho}_k^{-1}(V)$. It follows that

$$\mathcal{Z}_{k+1}(U_0, \dots, U_c) \cap (E^{\leq k+1} \setminus \mathcal{Z}_{k+1}(K_1, \dots, K_d, K)) \subseteq \tilde{\rho}_k^{-1}(V) \cap (E^{\leq k+1} \setminus K).$$

Hence, every open set in \mathcal{B} contains an element of the basis \mathcal{B}_k . The result now follows. \square

Using Proposition 3.3.9 we define a basis for the topology on $E^{\leq \infty}$. Suppose that

$c \in \mathbb{N}_0$ and for all $0 \leq m \leq c$, U_m is an open subset of E^m . Define

$$\begin{aligned} \mathcal{Z}(U_0, U_1, \dots, U_c) &:= \tilde{\rho}_{\infty, c}^{-1}(\mathcal{Z}_c(U_0, U_1, \dots, U_c)) \\ &= \bigcap_{m=0}^c \tilde{\rho}_{\infty, m}^{-1}(U_m) \\ &= \{x \in E^{\leq \infty} \mid x_{[0, m]} \in U_m \text{ for } 0 \leq m \leq c\} \end{aligned}$$

Similarly, suppose that $d \in \mathbb{N}_0$ and that for all $1 \leq n \leq d$, K_n is a compact subset of E^n . Define

$$\begin{aligned} \mathcal{Z}(K_1, K_2, \dots, K_d) &= \tilde{\rho}_{\infty, d}^{-1}(\mathcal{Z}_d(K_1, K_2, \dots, K_d)) \\ &= \bigcap_{n=1}^d \tilde{\rho}_{\infty, n}^{-1}(K_n) \\ &= \{x \in E^{\leq \infty} \mid x_{[0, n]} \in K_n \text{ for } 1 \leq n \leq d\}. \end{aligned}$$

Corollary 3.3.10. *The collection,*

$$\begin{aligned} \mathcal{B}_\infty &:= \{\mathcal{Z}(U_0, \dots, U_c) \cap (E^{\leq \infty} \setminus \mathcal{Z}(K_1, \dots, K_d)) \mid c \in \mathbb{N}_0, d \in \mathbb{N}, U_m \subseteq E^m \text{ open,} \\ &\quad \text{and } K_c \subseteq E^c \text{ compact}\} \end{aligned}$$

is a basis for the initial topology on $E^{\leq \infty}$.

Proof. This follows immediately from Proposition 3.3.9 and definition of the initial topology. \square

Since \mathcal{B}_∞ defines a basis for the topology on $E^{\leq \infty}$ it is now immediate to see that the topology on $E^{\leq \infty}$ agrees with the patch topology of [dCa18, Proposition 3.18]: that is the coarsest topology on $E^{\leq \infty}$ generated by cylinder sets $\mathcal{Z}(U_0, U_1, \dots, U_m)$ together with the cocompact topology it generates. Indeed, this implies that when E is a topological graph (cf. [Kat04a]) the topology on $E^{\leq \infty}$ that we have constructed agrees with those previously considered in [Web14; Yee07; PW05] (see [dCa18] for details).

Remark 3.3.11. By Remark 3.3.5, if $E_{rsing}^0 = \emptyset$, then the topology on $E^{\leq \infty}$ is the disjoint union topology.

3.3.1.2 The boundary path space ∂E

Although the construction of the path space $E^{\leq \infty}$ in the previous subsection gives an extension of E^∞ which is locally compact, the construction is by not especially efficient since there are smaller perfections than the unified space. In this subsection we instead apply the minimal perfection of Appendix C.3.2 inductively to the maps $\rho_n: E^{n+1} \rightarrow E^n$. The inverse limit of the induced system turns out to be the boundary path space ∂E in the case where E is a topological graph.

As a set, the minimal perfection of $r: E^1 \rightarrow E^0$ is the closed subset of the unified space $E^{\leq 1}$ given by

$$(E^1)_r^+ = E^1 \sqcup \left(E^0 \setminus \left(U \cap \text{int} \left(\overline{r(E^1)} \right) \right) \right) \subseteq E^1 \sqcup E^0,$$

where U is the largest open subset of E^0 such that $r|_{r^{-1}(U)}$ is proper (see (C.3)). The set $E^0 \setminus \text{int}(\overline{r(E^1)}) = \overline{E^0 \setminus r(E^1)}$ is the closure of the collection of sources $\overline{E_{src}^0}$ and we make this identification. Proposition C.3.13 implies that U can be identified with the set of finite receivers E_{fr}^0 , and so $E^0 \setminus U = E_{inf}^0$. As a result of these identifications, $(E^1)_r^+$ may be identified with the set $E^1 \sqcup (E_{inf}^0 \cup \overline{E_{src}^0}) = E^1 \sqcup E_{rsing}^0$. We denote the set $(E^1)_r^+$ by $E^{1,+}$ and the restriction of \tilde{r} to $E^{1,+}$ by $r^+ : E^{1,+} \rightarrow E^0$. Then r^+ is perfect and given by

$$r^+(x) = \begin{cases} r(x) & \text{if } x \in E^1 \\ x & \text{if } x \in E_{rsing}^0. \end{cases}$$

Remark 3.3.12. If $E_{rsing}^0 = \emptyset$, then $E^{1,+}$ is just E^1 .

Just as when $E^{\leq \infty}$ was constructed, we compose $\rho_1 : E^2 \rightarrow E^1$ with the open inclusion $E^1 \hookrightarrow E^{1,+}$ to arrive at a continuous map $\rho_1 : E^2 \rightarrow E^{1,+}$. We then define $E^{2,+} := (E^2)_{\rho_1}^+$ which has associated perfect map $\rho_1^+ : E^{2,+} \rightarrow E^{1,+}$. Note that $E^{2,+}$ can be identified as a closed subspace of $E^{\leq 2}$ since $E^{2,+} \subseteq E^2 \sqcup E^{1,+} \subseteq E^2 \sqcup E^{\leq 1}$. Continuing inductively, for each $n \in \mathbb{N}$ define $E^{n,+} := (E^n)_{\rho_{n-1}}^+$ with associated perfect maps $\rho_n^+ : E^{n+1,+} \rightarrow E^{n,+}$. For each $n \in \mathbb{N}$, $E^{n,+}$ is a closed subset of $E^{\leq n}$, and E^n is open in $E^{n,+}$.

Definition 3.3.13. Let E be a topological quiver. The *boundary path space* of E is defined to be the inverse limit space $\partial E := \varprojlim (E^{n,+}, \rho_n^+)$.

Corollary 3.3.3 implies that ∂E is a second-countable locally compact Hausdorff space. The universal maps $\rho_{\infty,n}^+ : \partial E \rightarrow E^{n,+}$ are also perfect.

To see that ∂E agrees with previous notions of boundary path spaces takes some effort. Recall that for each $n \in \mathbb{N}$, we have $E_{inf}^n = s_n^{-1}(E_{inf}^0)$ and $E_{src}^n = s_n^{-1}(E_{src}^0)$. Since s_n is an open map we have

$$\begin{aligned} E_{src}^n &= s_n^{-1}(E^0 \setminus \overline{r(E^1)}) = s_n^{-1}(\text{int}(E^0 \setminus r(E^1))) = \text{int}(s_n^{-1}(E^0 \setminus r(E^1))) \\ &= \text{int}(E^n \setminus s_n^{-1} \circ r(E^1)) = E^n \setminus \overline{s_n^{-1} \circ r(E^1)}. \end{aligned} \quad (3.25)$$

Define $E_{rsing}^n := E_{inf}^n \cup \overline{E_{src}^n}$, where the closure is taken in E^n . The spaces $E^{n,+}$ can be described in terms the sets E_{rsing}^k for $k \leq n$.

Proposition 3.3.14. For each $n \in \mathbb{N}$ we have $E^{n,+} = E^n \sqcup \bigsqcup_{k=0}^{n-1} E_{rsing}^k$ as sets.

Proof. We have already shown that $E^{1,+} = E^1 \cup E_{rsing}^0$. In general, it follows from (C.2) and Proposition C.3.13 that $E^{n+1,+} = E^{n+1} \sqcup ((E^{n,+} \setminus U_n) \cup \widetilde{\rho_n^{-1}}(E^{n,+} \setminus \text{int}(\overline{\rho_n(E^{n+1})})))$, where U_n is the largest open subset of $E^{n,+}$ such that for all compact $K \subseteq U_n$ the set $\rho_n^{-1}(K)$ is compact. Thus, our task is to identify $E^{n+1,+} \setminus E^{n+1}$. In particular, we must identify both $E^{n,+} \setminus U_n$ and $E^{n,+} \setminus \text{int}(\overline{\rho_n(E^{n+1})})$. To this end, suppose for induction that $E^{n,+} \setminus E^n = \bigsqcup_{k=0}^{n-1} E_{rsing}^k$. To describe $E^{n,+} \setminus U_n$ we split it into two parts $A := (E^{n,+} \setminus U_n) \cap E^n = E^n \setminus U_n$ and $B := (E^{n,+} \setminus U_n) \cap (E^{n,+} \setminus E^n)$, and describe these individually.

(A) We claim that $A = E_{inf}^n = s_n^{-1}(E_{inf}^0)$. We begin by showing that $s_n^{-1}(E_{fr}^0) \subseteq U_n$. Suppose that $K \subseteq E^{n,+}$ is a compact subset of $s_n^{-1}(E_{fr}^0) \subseteq E^n \subseteq E^{n,+}$. Then $s_n(K)$ is a compact subset of E_{fe}^0 so that $r^{-1}(s_n(K))$ is compact. It now follows that

$\rho_n^{-1}(K) = K \times_{s_n, r} r^{-1}(s_n(K))$ is compact in E^{n+1} . Hence, $s_n^{-1}(E_{fr}^0) \subseteq U_n$, and so $E^n \setminus U \subseteq s_n^{-1}(E_{inf}^0)$.

On the other hand, suppose that $y \in s_n^{-1}(E_{inf}^0)$ and fix a precompact neighbourhood W of y in E^n . Then $s_n(W)$ is a precompact open neighbourhood of $s_n(y)$ since s_n is open and $\overline{s_n(W)} \subseteq s_n(\overline{W})$. As $s_n(y) \in E_{inf}^0$, it follows from Proposition C.3.13 that $r^{-1}(s_n(\overline{W}))$ is not compact. In particular, $\rho_n^{-1}(\overline{W}) = \overline{W} \times_{s_n, r} r^{-1}(s_n(\overline{W}))$ is not compact in E^{n+1} . Consequently, $y \in E^n \setminus U$, giving $s_n^{-1}(E_{inf}^0) \subseteq E^n \setminus U$.

- (B) We claim that $B = \overline{\rho_n(E^{n+1})} \cap (E^{n,+} \setminus E^n)$. Since $\rho_n(E^{n+1}) \subseteq E^n$ it follows from Lemma C.3.16 that $\overline{\rho_n(E^{n+1})} \cap (E^{n,+} \setminus E^n)$ is a subset of $E^{n,+} \setminus U_n$. On the other hand, if $y \in (E^{n,+} \setminus E^n) \setminus \overline{\rho_n(E^{n+1})}$ then $W := (E^{n,+} \setminus E^n) \setminus \overline{\rho_n(E^{n+1})}$ is an open neighbourhood of y in $E^{n,+}$ with the property that $\rho_n^{-1}(W) = \emptyset$ is compact. Consequently, $y \in U_n$. Note that B is closed in $E^{n,+}$.

It follows that $E^{n,+} \setminus U_n = A \cup B = E_{inf}^n \cup (\overline{\rho_n(E^{n+1})} \cap (E^{n,+} \setminus E^n))$.

We now describe the set $E^{n,+} \setminus \text{int}(\overline{\rho_n(E^{n+1})}) = \overline{E^{n,+} \setminus \overline{\rho_n(E^{n+1})}}$. To do so we realise $E^{n,+} \setminus \overline{\rho_n(E^{n+1})}$ as the intersection of two sets $C := E^{n,+} \setminus (\rho_n(E^{n+1}) \cap E^n)$ and $D := E^{n,+} \setminus (\overline{\rho_n(E^{n+1})} \cap (E^{n,+} \setminus E^n))$, which we now describe.

- (C) Observe that $\rho_n(E^{n+1}) = s_n^{-1} \circ r(E^1)$. Thus, using (3.25),

$$C = E^{n,+} \setminus (\overline{s_n^{-1} \circ r(E^1)} \cap E^n) = (E^{n,+} \setminus E^n) \cup (E^n \setminus \overline{s_n^{-1} \circ r(E^1)}) = (E^{n,+} \setminus E^n) \cup E_{src}^n.$$

- (D) We have $D = E^n \cup ((E^{n,+} \setminus E^n) \setminus \overline{\rho_n(E_{n+1})})$.

Intersecting C and D we arrive at

$$E^{n,+} \setminus \overline{\rho_n(E^{n+1})} = C \cap D = E_{src}^n \cup ((E^{n,+} \setminus E^n) \setminus \overline{\rho_n(E_{n+1})}).$$

Taking closures now gives

$$E^{n,+} \setminus \text{int}(\overline{\rho_n(E^{n+1})}) = \overline{E^{n,+} \setminus \overline{\rho_n(E^{n+1})}} = \overline{E_{src}^n} \cup \overline{(E^{n,+} \setminus E^n) \setminus \overline{\rho_n(E_{n+1})}}.$$

Combining this with our characterisation of $E^{n,+} \setminus U_n$ we see that,

$$\begin{aligned} E^{n+1,+} \setminus E^{n+1} &= (E^{n,+} \setminus U_n) \cup \widetilde{\rho_n}^{-1}(E^{n,+} \setminus \text{int}(\overline{\rho_n(E^{n+1})})) \\ &= E_{inf}^n \cup (\overline{\rho_n(E^{n+1})} \cap (E^{n,+} \setminus E^n)) \cup \overline{E_{src}^n} \cup \overline{(E^{n,+} \setminus E^n) \setminus \overline{\rho_n(E_{n+1})}} \\ &= E_{rsing}^n \cup \overline{(\rho_n(E^{n+1}) \cap (E^{n,+} \setminus E^n))} \cup \overline{(E^{n,+} \setminus E^n) \setminus \overline{\rho_n(E_{n+1})}} \\ &= E_{rsing}^n \cup \overline{(E^{n,+} \setminus E^n)}, \end{aligned}$$

and so the inductive hypothesis gives

$$E^{n+1,+} \setminus E^{n+1} = E_{rsing}^n \sqcup \bigsqcup_{k=0}^{n-1} E_{rsing}^k. \quad \square$$

Remark 3.3.15. If $E_{rsing}^0 = \emptyset$, then Remark 3.3.12 shows that ∂E homeomorphic to the infinite path space E^∞ . In particular, if E is the topological quiver arising from an iterated

function system (\mathbb{A}, Γ) from Example 3.1.12, then $\partial E \simeq E^\infty$ can be identified with the attractor $\tilde{\mathbb{A}}$ of the inverse lifted system of Section 2.5.2.

By using perfections the boundary path space ∂E comes with the following characterisation. We note that this differs from the characterisation given in [Car+11, Proposition 5.16], which makes reference to an ambient groupoid.

Proposition 3.3.16. *The boundary path space ∂E can be identified with the closed subset $E^\infty \sqcup \bigsqcup_{k=0}^\infty E_{rsing}^k$ of $E^{\leq \infty}$. Under this identification, ∂E is the smallest closed subset C of $E^{\leq \infty}$ —in the inclusion ordering—containing E^∞ , such that the restrictions of $\tilde{\rho}_{\infty,n}: E^{\leq \infty} \rightarrow E^n$ to C are perfect for all $n \in \mathbb{N}_0$.*

Proof. The description of ∂E as a set follows from Proposition 3.3.14. As remarked upon previously, $E^{n,+}$ can be realised as a closed subset of $E^{\leq n}$. Let $\psi_n: E^{n,+} \rightarrow E^{\leq n}$ denote the closed inclusion. Then $\psi_n \circ \rho_{\infty,n}^+: \partial E \rightarrow E^{\leq n}$ is a continuous closed map satisfying $(\psi_n \circ \rho_{\infty,n}^+) = \tilde{\rho}_n \circ (\psi_{n-1} \circ \rho_{\infty,n-1}^+)$ for each $n \in \mathbb{N}$. In particular, the universal property of $E^{\leq \infty}$ as an inverse limit in the category of topological spaces with continuous maps, gives a unique continuous map $\psi: \partial E \rightarrow E^{\leq \infty}$ such that $\tilde{\rho}_{\infty,n} \circ \psi = \psi_n \circ \rho_{\infty,n}^+$ for all $n \in \mathbb{N}$. Injectivity of ψ follows from injectivity of ψ_n for each $n \in \mathbb{N}$. That ψ is closed follows from the fact that $\psi_n \circ \rho_{\infty,n}^+$ is closed for each $n \in \mathbb{N}$.

For the final statement suppose that C is a closed subset of $E^{\leq \infty}$ containing E^∞ , and that $\tilde{\varphi}_{\infty,n}|_C$ is perfect. Then $\tilde{\rho}_{\infty,n}(C)$ is a closed subset of $E^{\leq n}$ containing E^n and $\tilde{\rho}_{n-1}|_{\tilde{\rho}_{\infty,n}(C)}$ is perfect. It now follows from Proposition C.3.13 that there is a closed inclusion $\psi_n: E^{n,+} \rightarrow \tilde{\rho}_{\infty,n}(C)$. Since $C = \varprojlim (\tilde{\rho}_{\infty,n}(C), \tilde{\rho}_n|_{\tilde{\rho}_{\infty,n+1}(C)})$, the argument from the first half of this proof gives a closed inclusion $\psi_C: E^{n,+} \rightarrow C \subseteq E^{\leq n}$ which agrees with ψ . \square

In previous constructions of the boundary path space, ∂E has been defined as the closed subset $E^\infty \sqcup \bigsqcup_{k=0}^\infty E_{rsing}^k$ of the path space $E^{\leq \infty}$, which is then equipped with the relative topology it inherits from $E^{\leq \infty}$. Since we have shown that $E^{\leq \infty}$ agrees topologically with previous constructions of the path space, it follows from Proposition 3.3.16 that our construction of ∂E also agrees with previous constructions (cf. [dCa18; Web14; Yee07]). The second part of Proposition 3.3.16 explains why ∂E occurs naturally when considering path spaces in topological graphs.

Finally, we state without proof a result for general inverse systems in the category of locally compact Hausdorff spaces with continuous maps. The details of the proof follow from what we have already seen. In particular, the construction of $E^{\leq \infty}$ and ∂E ; and the proofs of Corollary 3.3.10 and Proposition 3.3.16.

Corollary 3.3.17. *Let $(X_n, \phi_n: X_{n+1} \rightarrow X_n)_{n \in \mathbb{N}_0}$ be an inverse system in the category of locally compact Hausdorff spaces with continuous maps, and let $\phi_{\infty,n}: \varprojlim (X_n, \phi_n) \rightarrow X_n$ denote the associated universal maps. Then $X_{\leq \infty} := \varprojlim (X_n, \phi_n) \sqcup \bigsqcup_{n=0}^\infty X_n$ admits a locally compact Hausdorff topology with basis,*

$$\left\{ \mathcal{Z}(U_0, U_1, \dots, U_c) \cap \mathcal{Z}(K_1, K_2, \dots, K_d)^c \mid c \in \mathbb{N}_0, d \in \mathbb{N}, U_m \text{ open in } X_m, K_n \text{ compact in } X_n \right\},$$

which makes the maps $\tilde{\phi}_{\infty,n}: X_{\leq\infty} \rightarrow X_n$ given by

$$\tilde{\phi}_{\infty,n}(x) = \begin{cases} x & \text{if } x \in \bigsqcup_{k=0}^n X_k \\ \phi_n \circ \cdots \circ \phi_{k-1}(x) & \text{if } x \in X_k \text{ for } k < n \\ \phi_{\infty,n}(x) & \text{if } x \in \varprojlim (X_n, \phi_n) \end{cases}$$

perfect. If X_{∂} denotes the smallest closed subset of $X_{\leq\infty}$ for which the restriction of $\tilde{\phi}_{\infty,n}$ to X_{∂} is perfect for all $n \in \mathbb{N}_0$, then X_{∂} can be is an inverse limit of locally compact Hausdorff spaces and perfect maps arising from an inductive application of the minimal perfection construction.

3.3.2 The étale case: topological graphs

We now return to the problem of reconstructing $\mathcal{O}_{X_E}^{\mathbb{T}}$ and $\mathcal{T}_{X_E}^{\mathbb{T}}$ on the level of groupoids. That is, we return to finding groupoid models for the algebras $B_{[0,n]}$ and $C_{[0,n]}$ of (3.6) and assembling them into a groupoid model for the core. Unfortunately, we have been unable to realise the algebras $B_{[0,n]}$ and $C_{[0,n]}$ as groupoid algebras in the case of general topological quivers. However, it is achievable in the case of topological graphs. In the process of constructing a groupoid model for $B_{[0,n]}$ and $C_{[0,n]}$ in the topological graph setting, we pinpoint where problems arise for general quivers.

For the remainder of this section, let $E = (E^0, E^1, r, s, \lambda)$ be a topological graph. So $s: E^1 \rightarrow E^0$ is a local homeomorphism, and λ_v is counting measure on $s^{-1}(v)$. Accordingly, the measures λ_v^n on E^n are counting measures on the source fibres. Lemma 3.2.7 implies that the associated groupoids R_n are all étale.

Although, in the étale case, groupoid models for $\mathcal{O}_{X_E}^{\mathbb{T}}$ and $\mathcal{T}_{X_E}^{\mathbb{T}}$ are already given by \mathcal{R}_E and \mathcal{TR}_E respectively, the novelty comes in the reconstruction of these groupoids from the groupoids R_n associated to $\text{End}_A^0(X_E^{\otimes n})$. In the process of reconstructing \mathcal{R}_E and \mathcal{TR}_E , we see precisely where difficulties arise in the non-étale setting. At present, we do not have a solution for these obstructions.

Also of note are the constructions we use to build \mathcal{R}_E and \mathcal{TR}_E , the first of which is the adjunction groupoid of Appendix C.2. As far as the author is aware this construction is original, though it is reminiscent of the factor groupoids considered by Putnam in [Put98]. The adjunction groupoid construction is likely to have applications beyond the current scope of our work, and will be the subject of future research.

The second construction we use is inverse limits of groupoids with respect to partial morphisms, which were introduced by Austin and Mitra in [AM18, Theorem 3.16]. These inverse limits allow us to assemble the groupoids associated to $B_{[0,n]}$ and $C_{[0,n]}$ into \mathcal{TR}_E and \mathcal{R}_E , respectively.

3.3.2.1 Reconstructing \mathcal{TR}_E

To begin we consider the simpler case of building $B_{[0,n]}$ and reconstructing \mathcal{TR}_E . The Cuntz-Pimsner case follows from similar considerations. The goal for this section is to inductively construct a series of étale groupoids G_k such that $C^*(G_k) \cong B_{[0,k]}$. By then taking inverse limits of the groupoids G_k in a suitable category, we recover the groupoid \mathcal{TR}_E of (3.5).

To motivate this construction, one should keep in mind Proposition 3.2.5. In the process of building G_k we make heavy use of the constructions presented in Appendix C, namely perfection and adjunction groupoids, frequently without comment. This is unavoidable without impenetrable notational.

Before we begin we recall from Definition 3.2.21 that for each $k \in \mathbb{N}_0$, the map $\rho_k: E^{k+1} \rightarrow E^k$ is given by,

$$\rho_n(x_1 \cdots x_{n+1}) = \begin{cases} x_1 \cdots x_n & \text{if } n \geq 1, \\ r(x) & \text{if } n = 0. \end{cases}$$

Construction: G_1 We begin by setting $G_0 := R_0 = E^0$. Comparing to the diagram (3.8) we want G_1 to give rise to a split-exact sequence

$$0 \longrightarrow C^*(R_1) \longrightarrow C^*(G_1) \overset{\curvearrowright}{\longrightarrow} C^*(G_0) \longrightarrow 0.$$

In particular, the groupoid G_1 must be built in such a way that allows functions in $C_c(R_1)$ —which are defined on equivalence classes of edges—to interact with functions in $C_c(E^0)$.

The range map $r: E^1 \rightarrow E^0$ provides a natural coupling between E^1 and E^0 . However, as we observed in Section 3.3.1, the induced map $r^*: C(E^0) \rightarrow C(E^1)$ does not take compactly supported functions to compactly supported functions. Moreover, functions with support in $E^0 \setminus r(E^1)$ lie in the kernel of r^* . To amend this, we again consider the unified space $E^{\leq 1}$ of E^1 with respect to $\rho_0 = r$ as we did in Section 3.3.1. The unified space comes equipped with a perfect map $\widetilde{\rho}_0: E^{\leq 1} \rightarrow E^0$. The dual $*$ -homomorphism $\widetilde{\rho}_0^*$ then takes functions in $C_c(E^0)$ to functions in $C_c(E^{\leq 1})$.

The issue now falls to constructing a groupoid G_1 for which $C_c(G_1)$ contains both $C_c(R_1)$ and $\widetilde{\rho}_0^*(C_c(R_0))$, and such that the product described in Lemma 3.2.2 is implemented by convolution on $C_c(G_1)$. The key to building G_1 is the adjunction groupoid construction of Theorem C.2.10. With this construction we “glue” R_1 to $E^{\leq 1}$ over the common subgroupoid E^1 .

It follows from Theorem C.3.6 that E^1 can be identified as an open subset of $E^{\leq 1}$. Since the unit space of R_1 is homeomorphic to E^1 , there is a continuous, open inclusion $\Sigma_1: R_1^{(0)} \hookrightarrow E^{\leq 1}$. As R_1 is étale, $R_1^{(0)}$ is clopen in R_1 . Hence, we can form the adjunction space,

$$G_1 := R_1 \sqcup_{\Sigma_1} E^{\leq 1} = (R_1 \sqcup E^{\leq 1}) / \{x \sim \Sigma_1(x) \mid x \in R_1^{(0)}\}. \quad (3.26)$$

Theorem C.2.10 implies that G_1 can be given the structure of an étale groupoid. As mentioned at the beginning of Appendix C.2, as a set G_1 can be explicitly decomposed as,

$$G_1 = R_1 \sqcup (E^{\leq 1} \setminus \Sigma_1(R_1^{(0)})) = R_1 \sqcup R_0. \quad (3.27)$$

The topology on G_1 is typically distinct from the disjoint union topology due to the use of both the unified space and adjunction space constructions.

Letting $i_1: R_1 \rightarrow G_1$ and $j_1: E^{\leq 1} \rightarrow G_1$ denote canonical inclusions (see Appendix C.2) the unit space $G_1^{(0)}$ is given by $i_1(R_1^{(0)}) \cup j_1(E^{\leq 1}) = j_1(E^{\leq 1})$, which is homeomorphic to $E^{\leq 1}$ as j_1 is open and injective. It also follows from Theorem C.2.10 that $i_1(R_1)^{(0)}$ is a G_1 -invariant subgroupoid of $G_1^{(0)}$. Since $E^{\leq 1}$ and R_1 are both amenable

groupoids it follows from Corollary C.2.14 that G_1 is also amenable. In Theorem 3.3.27 we show that $C^*(G_1)$ is isomorphic to $B_{[0,1]}$. \triangleleft

Remark 3.3.18. Before we move on, we remark that we have already used the hypothesis that R_1 is étale in a critical way. Suppose that E were instead a topological quiver. Then $R^{(0)}$ remains a closed subgroupoid of R_1 ; $\Sigma_1: R_1^{(0)} \rightarrow E^{\leq 1}$ is a continuous, open, injective, groupoid homomorphism; and $\Sigma_1(R_1^{(0)}) \simeq E^1$ is trivially $E^{\leq 1}$ -invariant.

However, it is not necessarily the case that for each $x \in E^0 = E^{\leq 1} \setminus E^1$ there exists an open neighbourhood $V \subseteq E^{\leq 1}$ of x such that $\Sigma_1^{-1}(V)$ is open in R_1 . Accordingly, the hypotheses of Lemma C.2.9 and Theorem C.2.10 do not necessarily apply. Of course this problem could be circumvented by imposing additional hypotheses. For example stipulating that E_{br}^1 is compact suffices, for then $E^{\leq 1} \setminus E_{br}^1$ is an open neighbourhood of E^0 in $E^{\leq 1}$ satisfying the hypotheses of Lemma C.2.9. The set E_{br}^1 is always compact in the case where E arises from an iterated function system as in Example 3.1.12.

Given that E_{br}^1 is compact, Theorem C.2.10 implies $G_1 := R_1 \sqcup_{\Sigma_1} E^{\leq 1}$ is a topological groupoid. The pressing issue now is that it is not clear how to endow G_1 with a Haar system which is simultaneously compatible with the Haar systems on both R_1 and R_0 . This problem is amplified for the higher groupoids G_k once they are built. At present there is no solution to this problem.

We now proceed to inductively construct G_k for $k \geq 1$

Construction: G_k The construction of G_k from G_{k-1} is slightly more elaborate than the construction of G_1 . Again it involves a two-step process: we first identify an action of G_k on $E^{\leq k+1}$ and build the corresponding transformation groupoid $G_k \ltimes E^{\leq k+1}$, then we form an adjunction groupoid from R_{k+1} and $G_k \ltimes E^{\leq k+1}$ by gluing over E^{k+1} . For the sake of clarity, we outline how to construct G_2 from G_1 . The general case follows the same line of argument. We summarise the results in Proposition 3.3.23.

To begin consider the map $\rho_1: E^2 \rightarrow E^1$ composed with the open inclusion of E^1 into $G_1^{(0)} \simeq E^{\leq 1}$, which we again denote by ρ_1 . The groupoid G_1 acts canonically on its unit space $G_1^{(0)} \simeq E^{\leq 1}$. The construction of G_2 consists of two steps:

- (1) lift the action of G_1 on $E^{\leq 1}$ to $E^{\leq 2}$, then
- (2) glue $G_1 \ltimes E^{\leq 2}$ to R^2 along a subgroupoid.

(1) Lifting the action of G_1 on $E^{\leq 1}$ to $E^{\leq 2}$ To lift the action of G_1 to $E^{\leq 2}$ we make use of the action of R_1 on E^2 given by Lemma 3.2.23, which has moment map $\rho_1: E^2 \rightarrow E^1$, and satisfies $(x, y) \cdot yz = xz$. It is worth noting that elements $(x, y) \in R_1$ with $s(x) = s(y) \in E_{src}^0$ play no role in the action of R_1 on E^2 : if $s(y) \in E_{src}^0$ then y is not in the range of the moment map ρ_1 .

The action of R_1 on E^2 extends to an action of G_1 on E^2 by extending the moment map $\rho_1: E^2 \rightarrow R_1^{(0)} \hookrightarrow G_1^{(0)}$. Since elements of $G_1^{(0)} \setminus R_1^{(0)}$ do not lie in the range of $\rho_1: E^2 \rightarrow G_1^{(0)}$, the formula (3.17) still applies. Since R_1 is open in G_1 , the new G_1 -action is continuous. Thus, E^2 is a G_1 -space. However, elements of R_0 and elements $(x, y) \in R_1$ (thought of as elements of G_1) with $s(y) \in E_{src}^0$ play no role in the action of G_1 on E^2 . Consequently, we can identify the transformation groupoid $R_1 \ltimes E^2$ with $G_1 \ltimes E^2$.

To incorporate the actions of both R_0 and elements $(x, y) \in R_1$ with $s(x) \in E_{src}^0$ on both E^1 and E^0 we consider the unified space $E^{\leq 2}$ of E^2 with respect to $\rho_1: E^2 \rightarrow E^{\leq 1}$.

Recall that ρ_1 has a perfect extension $\widetilde{\rho}_1: E^{\leq 2} \rightarrow E^{\leq 1}$. Then $E^{\leq 2} = E^2 \sqcup E^{\leq 1}$ as a set, with E^2 open in $E^{\leq 2}$ (see Section 3.3.1). Since $\rho_1: E^2 \rightarrow G_1^{(0)}$ is a moment map, it is G_1 -equivariant. Applying Proposition C.3.17, we see that $E^{\leq 2}$ is a G_1 -space with moment map $\widetilde{\rho}_1: E^{\leq 2} \rightarrow G_1^{(0)}$ and left action $\cdot: G_1 \times_{s, \widetilde{\rho}_1} E^{\leq 2} \rightarrow E^{\leq 2}$ given by

$$\gamma \cdot z = \begin{cases} xz_2 & \text{if } \gamma = (x, y) \in R_1, z \in E^2, \text{ and } y = z_1, \\ r_{G_1}(\gamma) & \text{if } z = s_{G_1}(\gamma) \in E^{\leq 1} \simeq G_1^{(0)}. \end{cases}$$

Tautologically, $\widetilde{\rho}_1: E^{\leq 2} \rightarrow E^{\leq 1}$ is G_1 -equivariant.

We now invoke Lemma C.3.20 to induce a perfect groupoid homomorphism between corresponding transformation groupoids $\widetilde{\rho}_1^\times: G_1 \ltimes E^{\leq 2} \rightarrow G_1 \cong G_1 \ltimes G_1^{(0)}$ satisfying

$$\widetilde{\rho}_1^\times(\gamma, z) = \gamma.$$

The groupoid $G_1 \ltimes E^{\leq 2}$ is étale since $s(\gamma, x) = (s(\gamma), x)$ for all $(\gamma, x) \in G_1 \ltimes E^{\leq 2}$ and G_1 is étale. It also follows from Lemma B.4.5 that $G_1 \ltimes E^{\leq 2}$ is amenable.

Remark 3.3.19. The groupoid $G_1 \ltimes E^{\leq 2}$ plays the role played by $E^{\leq 1}$ in the construction of G^1 . Indeed, the construction of G_1 is the degenerate case: $E^{\leq 1}$ is isomorphic as a topological groupoid to $G_0 \ltimes E^{\leq 1} = E^0 \ltimes E^{\leq 1}$, where $x \in E^0$ acts on $y \in E^{\leq 1}$ if $r(y) = x$, in which case $x \cdot y = y$.

We note that E^2 is an open G_1 -invariant subset of $E^{\leq 2}$. Consequently, $R_1 \ltimes E^2 = G_1 \ltimes E^2$ can be identified as an open subgroupoid of $G_1 \ltimes E^{\leq 2}$. As sets,

$$G_1 \ltimes E^{\leq 2} = (R_1 \ltimes E^2) \sqcup (G_1 \ltimes E^{\leq 1}) = (R_1 \ltimes E^2) \sqcup G_1 = (R_1 \ltimes E^2) \sqcup R_1 \sqcup R_0. \quad (3.28)$$

By Theorem B.2.9, we have a short exact sequence

$$0 \longrightarrow C^*(R_1 \ltimes E^2) \longrightarrow C^*(G_1 \ltimes E^{\leq 2}) \longrightarrow C^*(G_1) \longrightarrow 0.$$

(2) Gluing $G_1 \ltimes E^{\leq 2}$ to R_2 We now turn to constructing G_2 by adjoining $G_1 \ltimes E^{\leq 2}$ to R_2 over E^2 . For each $k \in \mathbb{N}$ we introduce a subgroupoid H_k of R_k defined by

$$H_k := \{(x, y) \in R_k \mid x_k = y_k\}. \quad (3.29)$$

In the case where $k = 1$ we have $H_1 = R_1^{(0)}$. The groupoid H_k is analogous to S_k of Section 3.2.3. To see that H_k is clopen in R_k , we again use the étale assumption.

Lemma 3.3.20. *Let E be a topological quiver. For each $k \in \mathbb{N}$ the subgroupoid H_k of R_k is closed. Moreover, if R_1 is étale (equivalently E is a topological graph) then H_k is also open in R_k .*

Proof. Fix a convergent net $(x^i, y^i) \rightarrow (x, y)$ in R_k with each $(x^i, y^i) \in H_k$. Then $x^i \rightarrow x$ and $y^i \rightarrow y$ in E^k . Since the map $z_1 \cdots z_k \mapsto z_k$ from E^k to E^1 is continuous, it follows that $x_k = y_k$, so $(x, y) \in H_k$. Hence, H_k is closed.

Now suppose that R_1 is étale and fix a convergent net $(x^i, y^i) \rightarrow (x, y)$ in R_k with each $(x^i, y^i) \in R_k \setminus H_k$. In particular, $x_k^i \neq y_k^i$ for all i . By continuity of the map

$z_1 \cdots z_k \mapsto z_k$, we have $x_k^i \rightarrow x_k$ and $y_k^i \rightarrow y_k$ in E^1 . Since $s(x_k^i) = s(y_k^i)$ for all i , and $s(x_k) = s(y_k)$ it follows that $(x_k^i, y_k^i) \rightarrow (x_k, y_k)$ in R_1 . As R_1 is étale we have $x_k \neq y_k$. Thus, $(x, y) \in R_k \setminus H_k$. Hence, H_k is open. \square

We also have the following analogue of Lemma 3.2.25.

Lemma 3.3.21. *Let E be a topological quiver and let $k \in \mathbb{N}$. Consider E^k as an R_{k-1} -space as in Lemma 3.2.23. Then there is an isomorphism of topological groupoids $\Sigma_k: H_k \rightarrow R_{k-1} \ltimes E^k$ satisfying,*

$$\Sigma_k(x, y) = ((x_{[0, k-1]}, y_{[0, k-1]}), y).$$

Proof. The map Σ_k is clearly a groupoid homomorphism. The inverse of Σ_k is given by $\Sigma_k^{-1}((x, y), z) = (xz_2, yz_2)$. Continuity of both Σ_k and Σ_k^{-1} is straightforward to show, and follows arguments similar to the proof of Lemma 3.2.23. \square

Remark 3.3.22. Although Lemma 3.3.21 is fairly innocuous, it is actually at this step where we hit another roadblock in the non-étale setting. Although $\Sigma_k: H_k \rightarrow R_{k-1} \ltimes E^k$ is a topological groupoid isomorphism, it does not typically preserve Haar systems. Indeed, if we equip H_k with the Haar system it inherits from R_k as a closed invariant subgroupoid, and equip $R_{k-1} \ltimes E^k$ with the Haar system inherited from R_{k-1} by Lemma B.4.4, then a calculation similar to (3.20) shows that a factor of $\lambda_{s(x_k)}(\{x_k\})$ appears. Previously, we relied on Lemma 3.2.26 to normalise (see Equation (3.22)) in order to induce an isomorphism between $C^*(R_{k-1}^{reg} \ltimes F^k)$ and $C^*(S_k)$. However, the existence of branched edges in E^1 forbids us from employing Lemma 3.2.26 to build an isomorphism from $C^*(R_{k-1} \ltimes E^k)$ to $C^*(H_k)$.

To save on notation we also denote the composition of $\Sigma_2: H_2 \rightarrow R_1 \ltimes E^2$ with the open inclusion of $R_1 \ltimes E^2$ into $G_1 \ltimes E^{\leq 2}$ by Σ_2 . Consequently, $\Sigma_2: H_2 \rightarrow G_1 \ltimes E^{\leq 2}$ is a continuous, open, injective, groupoid homomorphism; and $\Sigma_2(H_2^{(0)}) \simeq E^2$ is G_1 -invariant. Since H_2 is clopen in R_2 by Lemma 3.3.20, and both R_2 and G_1 are étale, we can apply Theorem C.2.10 to arrive at an étale groupoid

$$G_2 := R_2 \sqcup_{\Sigma_2} (G_1 \ltimes E^{\leq 2})$$

which has unit space $E^{\leq 2}$. Since R_2 and $G_1 \ltimes E^{\leq 2}$ are amenable, so is G_2 . Following on from Equation 3.28, as sets,

$$G_2 = R_2 \sqcup R_1 \sqcup R_0.$$

Nothing in the preceding construction of G_2 is special to the $k = 2$ case. We can repeat the same process inductively to define amenable étale groupoids G_k from G_{k-1} and R_k for all $k \in \mathbb{N}$. \triangle

To summarise the above constructions in the general case we record the following.

Proposition 3.3.23. *Let E be a topological graph. For each $k \in \mathbb{N}$ there are amenable étale groupoids defined by setting $G_0 = R_0$ and inductively defining,*

$$G_k := R_k \sqcup_{\Sigma_k} (G_{k-1} \ltimes E^{\leq k}),$$

where $\Sigma_k: H_k \rightarrow R_{k-1} \ltimes E^k$ is the isomorphism of Lemma 3.3.21. Moreover, G_k has the following properties:

- (i) the unit space $G_k^{(0)}$ is homeomorphic to $E^{\leq k}$;
- (ii) E^k is an open G_k -invariant subset of $G_k^{(0)}$, and $G_k|_{E^k}$ can be identified with R_k as a topological groupoid;
- (iii) $E^{\leq k-1} = E^{\leq k} \setminus E^k$ is a closed G_k -invariant subset of $G_k^{(0)}$, and $G_k|_{E^{\leq k-1}}$ can be identified with G_{k-1} as a topological groupoid;
- (iv) as sets we have a decomposition $G_k = R_k \sqcup G_{k-1} = \bigsqcup_{i=0}^k R_i$, so that

$$G_k = \{(x, y) \in E^{\leq k} \times E^{\leq k} \mid \ell(x) = \ell(y) \text{ and } s(x) = s(y)\}; \quad (3.30)$$

- (v) $G_k \ltimes E^{\leq k+1}$ can be identified as an open subgroupoid of G_{k+1} , and as sets

$$G_k \ltimes E^{\leq k+1} = H_{k+1} \sqcup G_k = H_{k+1} \sqcup \bigsqcup_{i=0}^k R_i; \text{ and}$$

- (vi) there is a perfect groupoid homomorphism $\widetilde{\rho}_k^\times: G_k \ltimes E^{\leq k+1} \rightarrow G_k$ which satisfies $\widetilde{\rho}_k^\times(\gamma, x) = \gamma$, and upon identifying $G_k \ltimes E^{\leq k+1}$ with $H_{k+1} \sqcup \bigsqcup_{i=0}^k R_i$ satisfies,

$$\widetilde{\rho}_k^\times(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in \bigsqcup_{i=0}^k R_i \\ (x_{[0,k]}, y_{[0,k]}) & \text{if } (x, y) \in H_{k+1}. \end{cases} \quad (3.31)$$

Proof. Each of the properties of G_k is evident from the inductive construction of G_k outlined prior to the statement of the proposition. The identification of R_k with $G_k|_{E^{\leq k-1}}$ is via the open inclusion $i_k: R_k \hookrightarrow G_k$ and the identification of $G_k \ltimes E^{\leq k+1}$ as an open subgroupoid of G_{k+1} is via the open inclusion $j_k: G_k \ltimes E^{\leq k+1} \hookrightarrow G_{k+1}$. Both come from the construction of the adjunction groupoid (cf. Theorem C.2.10). \square

From now on we identify $C_c(R_k)$ as a $*$ -subalgebra of $C_c(G_k)$ and $C^*(R_k)$ as a C^* -subalgebra of $C^*(G_k)$. Our task is now to verify that $C^*(G_k)$ is actually isomorphic to $B_{[0,k]}$. To do this we make use of Proposition 3.2.5. We begin with the following version of the split exact sequence (3.8).

Corollary 3.3.24. *For each $k \in \mathbb{N}_0$ there is a split exact sequence*

$$\begin{array}{ccccccc} & & C^*(G_k \ltimes E^{\leq k+1}) & & & & \\ & & \downarrow \widetilde{\beta}_k & \swarrow \widetilde{\alpha}_k & & & \\ 0 & \longrightarrow & C^*(R_{k+1}) & \longrightarrow & C^*(G_{k+1}) & \longrightarrow & C^*(G_k) \longrightarrow 0. \end{array} \quad (3.32)$$

where $\widetilde{\alpha}_k$ is induced by the perfect homomorphism $\rho_k^\times: G_k \ltimes E^{\leq k+1} \rightarrow G_k$, and $\widetilde{\beta}_k$ is induced by the extension by zero of functions in $C_c(G_k \ltimes E^{\leq k+1})$ to $C_c(G_{k+1})$.

Proof. For each $k \in \mathbb{N} \sqcup \{0\}$ we have the decomposition $G_k \ltimes E^{\leq k+1} = (R_k \ltimes E^{k+1}) \sqcup G_k$ as sets, and $R_k \ltimes E^{k+1}$ is open in $G_k \ltimes E^{\leq k+1}$. It follows from Corollary C.2.15 that for

each $k \in \mathbb{N}$ we have a commuting diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & C^*(R_k \rtimes E^{k+1}) & \hookrightarrow & C^*(G_k \rtimes E^{\leq k+1}) & \longrightarrow & C^*(G_k) \longrightarrow 0 \\
& & \downarrow (\Sigma_k^{-1})^* & & \downarrow \widetilde{\beta}_k & & \parallel \\
0 & \longrightarrow & C^*(R_{k+1}) & \hookrightarrow & C^*(G_{k+1}) & \longrightarrow & C^*(G_k) \longrightarrow 0
\end{array}$$

with exact rows.

Since $\widetilde{\rho}_k^{\times}: G_k \rtimes E^{\leq k+1} \rightarrow G_k$ is a perfect groupoid homomorphism between étale groupoids, it follows from Proposition B.2.7 that it induces an injective $*$ -homomorphism $\widetilde{\alpha}_k: C^*(G_k) \rightarrow C^*(G_k \rtimes E^{\leq k+1})$ such that $\widetilde{\alpha}_k(f)(\gamma, x) = f(\gamma)$ for all $f \in C_c(G_k)$. Consequently, for each $f \in C_c(G_k)$ we have

$$\widetilde{\beta}_k \circ \widetilde{\alpha}_k(f)(\gamma) = \begin{cases} f(\gamma') & \text{if } \gamma = (\gamma', x) \in G_k \rtimes E^{\leq k+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Since the quotient in (3.32) is induced by restriction of functions to G_k , it follows that $\widetilde{\beta}_k \circ \widetilde{\alpha}_k$ is a splitting of the quotient map. \square

To compactify notation, for each $k \in \mathbb{N}$ define $\widetilde{\tau}_k := \widetilde{\beta}_k \circ \widetilde{\alpha}_k: C^*(G_k) \rightarrow C^*(G_{k+1})$. With the identification of G_k given by (3.30), $\widetilde{\tau}_k$ satisfies

$$\widetilde{\tau}_k(f)(x, y) = \begin{cases} f(x, y) & \text{if } m \leq k; \\ f(x_{[0, k]}, y_{[0, k]}) & \text{if } m = k + 1 \text{ and } x_{k+1} = y_{k+1}; \\ 0 & \text{otherwise,} \end{cases}$$

for all $f \in C_c(G_k)$ and $(x, y) \in R_m$, where $0 \leq m \leq k + 1$.

For $0 \leq l < k$ define $\widetilde{\tau}_{[l, k]} := \widetilde{\tau}_k \circ \cdots \circ \widetilde{\tau}_l: C^*(G_l) \rightarrow C^*(G_k)$, and let $\widetilde{\tau}_{[k, k]} = \text{id}_{C^*(G_k)}$. Using the description of $\widetilde{\tau}_k$ above we see that

$$\widetilde{\tau}_{[l, k]}(f)(x, y) = \begin{cases} f(x, y) & \text{if } m \leq l; \\ f(x_{[0, l]}, y_{[0, l]}) & \text{if } l < m \leq k \text{ and } x_{[l+1, m]} = y_{[l+1, m]}; \\ 0 & \text{otherwise;} \end{cases}$$

for all $f \in C_c(G_l)$ and $(x, y) \in R_m$, where $0 \leq m \leq k$.

Lemma 3.3.25. *For each $a \in C^*(G_k)$ there exist $a_0 \in C^*(R_0), \dots, a_k \in C^*(R_k)$ for all $0 \leq l \leq k$ such that $a = \sum_{l=0}^k \widetilde{\tau}_{[l, k]}(a_l)$.*

Proof. This follows from inductive application of the Splitting Lemma (Corollary D.0.3) to the split exact sequence (3.32). Indeed, there exist $a_l \in R_l$ and $b_l \in G_l$ such that

$$a = a_k + \widetilde{\tau}_k(b_{k-1}) = a_k + \widetilde{\tau}_k(a_{k-1} + \widetilde{\tau}_{k-1}(b_{k-2})) = \cdots = \sum_{l=0}^k \widetilde{\tau}_{[l, k]}(a_l). \quad \square$$

The next step is to describe multiplication in $C^*(G_k)$. By Lemma 3.3.25 it suffices to know how to multiply elements $a \in C^*(R_k) \subseteq C^*(G_k)$ with $\tilde{\tau}_{[m,k]}(b)$ for $b \in C^*(R_m)$, where $m \leq k$.

Lemma 3.3.26. *Let $k \in \mathbb{N}_0$ and $m \leq k$. Then for all $a \in C^*(R_k)$ and $b \in C^*(R_m)$ we have*

$$a\tilde{\tau}_{[m,k]}(b) = a *_{k,m} b \quad \text{and} \quad \tilde{\tau}_{[m,k]}(b)a = b *_{m,k} a,$$

where $*_{k,m}: C^*(R_k) \times C^*(R_m) \rightarrow C^*(R_k)$ is as defined in Proposition 3.2.20.

Proof. If $m = k$ then both sides correspond to the usual product on $C^*(R_k)$. So suppose that $f \in C_c(R_k)$ and $g \in C_c(R_m)$ for $m < k$. Then for $(x, y) \in G_k = \bigsqcup_{m=0}^k R_m$ we have,

$$\begin{aligned} (f * \tilde{\tau}_{[m,k]}(g))(x, y) &= \begin{cases} \sum_{(x,z) \in R_k} f(x, z) \tilde{\tau}_{[m,k]}(g)(z, y) & \text{if } (x, y) \in R_k, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \sum_{(z', y_{[0,m]}) \in R_m} f(x, z' y_{[m+1,k]}) g(z', y_{[0,m]}) & \text{if } (x, y) \in R_k, \\ 0 & \text{otherwise,} \end{cases} \\ &= (f *_{k,m} g)(x, y). \end{aligned}$$

Since $C_c(R_k) \times C_c(R_m)$ is dense in $C^*(R_k) \times C^*(R_m)$, it follows from continuity (using the inequality $\|a *_{k,m} b\| \leq \|a\| \|b\|$) that $a\tilde{\tau}_{[m,k]}(b) = a *_{k,m} b$ for all $a \in C^*(R_k)$ and $b \in C^*(R_m)$. A symmetric argument shows that $\tilde{\tau}_{[m,k]}(b)a = b *_{m,k} a$. \square

We can now prove that G_k does indeed give us a groupoid model for $B_{[0,k]}$.

Theorem 3.3.27. *Let E be a topological graph. For each $\ell \in \mathbb{N}_0$ let $\Phi_\ell: C^*(R_\ell) \rightarrow \text{End}_A^0(X_E^{\otimes \ell})$ be the isomorphism of Corollary 3.2.17. Then for each $k \in \mathbb{N}$ there is an isomorphism $\Upsilon_k: C^*(G_k) \rightarrow B_{[0,k]}$ of C^* -algebras, where $\Upsilon_0 = j_A$, and for all $k \in \mathbb{N}_0$ and all $l \leq k$ the diagram*

$$\begin{array}{ccc} C^*(R_l) & \xrightarrow{\Phi_l} & \text{End}_A^0(X_E^{\otimes l}) \\ \tilde{\tau}_{[l,k]} \downarrow & & \downarrow j_{X_E}^{(l)} \\ C^*(G_k) & \xrightarrow{\Upsilon_k} & B_{[0,k]} \end{array} \quad (3.33)$$

commutes.

Proof. It follows from Lemma 3.3.26 and Proposition 3.2.20 that $(C^*(G_k), \tilde{\tau}_k)_{k \in \mathbb{N}_0}$ satisfies condition (iii) of Proposition 3.2.5. Condition (ii) of Proposition 3.2.5 follows from Corollary 3.3.24, and condition (i) is satisfied since $C^*(R_0) \cong C_0(E^0)$. The result now follows from Proposition 3.2.5. \square

For each $k \in \mathbb{N}_0$, let $\tilde{\tau}_{[k,\infty]}: C^*(G_k) \rightarrow \varinjlim (C^*(G_k), \tilde{\tau}_k)$ and $\iota_{k,\infty}: B_{[0,k]} \rightarrow \mathcal{T}_{X_E}^\mathbb{T}$ denote the universal inclusions. Proposition 3.2.1 gives the following.

Corollary 3.3.28. *There is an isomorphism $\Upsilon: \varinjlim (C^*(G_k), \tilde{\tau}_k) \rightarrow \mathcal{T}_E^\mathbb{T}$ such that $\Upsilon \circ \tilde{\tau}_{[k,\infty]} = \iota_{k,\infty} \circ \Upsilon_k$ for all $k \in \mathbb{N}_0$.*

Armed with a groupoid model for $B_{[0,k]}$, our final task is to recover the groupoid \mathcal{TR}_E from the groupoids $\{G_k\}_{k \in \mathbb{N}_0}$. To do so requires the notion of partial morphisms.

Definition 3.3.29 (Austin-Mitra [AM18]). Let G and H be topological groupoids with Haar systems. A *partial morphism* from G to H is a pair (f, K) consisting of:

- (i) an open subgroupoid K of G with Haar system inherited from G ; and
- (ii) a Haar-system-preserving proper groupoid homomorphism $f: K \rightarrow H$.

We write $(f, K): G \rightarrow H$ to mean that (f, K) is a partial morphism from G to H .

It follows from a combination of Proposition B.2.7 and Proposition B.2.8 that a partial morphism $(f, K): G \rightarrow H$ induces a $*$ -homomorphism from $C_c(H) \xrightarrow{f^*} C_c(K) \hookrightarrow C_c(G)$ which extends to both the full and reduced C^* -completions. We only deal with partial morphisms where $f: K \rightarrow H$ is also surjective. In this case the induced $*$ -homomorphism is isometric (see [AM18, Remark 3.7])

In [AM18, Theorem 3.16] Austin and Mitra show that second-countable locally compact Hausdorff groupoids with Haar systems, together with partial morphisms between them, form a category. Moreover, every inverse system in this category has an inverse limit. Such an inverse system induces a directed system of groupoid C^* -algebras. It follows from [AM18, Theorem 3.19] that the groupoid C^* -algebra of the inverse limit groupoid is isomorphic to the direct limit of the groupoid C^* -algebras.

Looking back to the construction of G_k , we see that $(\widetilde{\rho}_k^\times, G_k \ltimes E^{\leq k+1})$ defines a partial morphism from G_{k+1} to G_k for each $k \in \mathbb{N}_0$. The induced map on C^* -algebras is precisely $\widetilde{\tau}_k: C^*(G_k) \rightarrow C^*(G_{k+1})$. Our claim is that the inverse limit groupoid of the inverse system $(G_k, (\widetilde{\rho}_k^\times, G_k \ltimes E^{\leq k+1}))_{k \in \mathbb{N}}$ is \mathcal{TR}_E . We describe the inverse limit using the construction outlined in the proof of [AM18, Theorem 3.19].

Construction: $\varprojlim (G_k, (\widetilde{\rho}_k^\times, G_k \ltimes E^{\leq k+1}))$ For each $n \in \mathbb{N}_0$ and each $k > n$ let

$$G_{n,k} := (\widetilde{\rho}_{k-1}^\times)^{-1} \circ \cdots \circ (\widetilde{\rho}_n^\times)^{-1}(G_n) \subseteq G_k, \quad (3.34)$$

and $G_{n,n} = G_n$. By construction, for each $k > n$, the restricted map $\widetilde{\rho}_k^\times: G_{n,k+1} \rightarrow G_{n,k}$ is a perfect groupoid homomorphism. It now makes sense to define $\widetilde{\rho}_{[n,k]}^\times: G_{n,k} \rightarrow G_n$ by

$$\widetilde{\rho}_{[n,k]}^\times := \widetilde{\rho}_n^\times \circ \cdots \circ \widetilde{\rho}_{k-1}^\times. \quad (3.35)$$

Consider the inverse limit,

$$G_{n,\infty} := \varprojlim (G_{n,k}, \widetilde{\rho}_k^\times).$$

Then $G_{n,\infty}$ is a second-countable locally compact Hausdorff étale groupoid by [AM18, Proposition 3.17]. When $n = 0$, we can identify $G_{0,k}$ with $E^{\leq k}$, in which case $\widetilde{\rho}_k^\times|_{E^{\leq k}}$ is just the map $\widetilde{p}_k: E^{\leq k+1} \rightarrow E^{\leq k}$. It follows from Section 3.3.1.1 that $G_{0,\infty}$ can be identified with the infinite path space $E^{\leq \infty}$.

For each $n \in \mathbb{N}_0$, we have $G_{n,n+1} = G_n \ltimes E^{\leq n+1}$, which is open in G_{n+1} . Consequently, $G_{n,k}$ is open in $G_{n+1,k}$ for all $n \in \mathbb{N}_0$ and $k \geq n+1$. It follows that there is an open inclusion $\iota_n: G_{n,\infty} \hookrightarrow G_{n+1,\infty}$ for each $n \in \mathbb{N}_0$. Hence, we can form the direct limit,

$$G_\infty = \varinjlim (G_{n,\infty}, \iota_n) = \bigcup_{n=1}^{\infty} G_{n,\infty},$$

which we equip with the final topology induced by the ι_n (in particular each $G_{n,\infty}$ is open in G_∞). The construction of G_∞ can be summarised with the following commuting diagram:

$$\begin{array}{ccccccc}
 G_0 & \xleftarrow{(\tilde{\rho}_0^\times, E^{\leq 1})} & G_1 & \xleftarrow{(\tilde{\rho}_1^\times, G_1 \times E^{\leq 2})} & G_2 & \xleftarrow{(\tilde{\rho}_2^\times, G_2 \times E^{\leq 3})} & \dots \\
 \tilde{\rho}_0^\times \uparrow & & \tilde{\rho}_1^\times \uparrow & & \tilde{\rho}_2^\times \uparrow & & \\
 E^{\leq 1} & & G_{1,2} = G_1 \times E^{\leq 2} & & G_{2,3} = G_2 \times E^{\leq 3} & & \dots \\
 \tilde{\rho}_1^\times \uparrow & & \tilde{\rho}_2^\times \uparrow & & \tilde{\rho}_3^\times \uparrow & & \\
 E^{\leq 2} & & G_{1,3} & & G_{2,4} & & \dots \\
 \tilde{\rho}_2^\times \uparrow & & \tilde{\rho}_3^\times \uparrow & & \tilde{\rho}_4^\times \uparrow & & \\
 E^{\leq 3} & & G_{1,4} & & G_{2,5} & & \dots \\
 \tilde{\rho}_3^\times \uparrow & & \tilde{\rho}_4^\times \uparrow & & \tilde{\rho}_5^\times \uparrow & & \\
 \vdots & & \vdots & & \vdots & & \\
 & & & & & & \\
 E^{\leq \infty} & \xrightarrow{\iota_0} & G_{1,\infty} & \xrightarrow{\iota_1} & G_{2,\infty} & \xrightarrow{\iota_2} & \dots \quad G_\infty,
 \end{array} \tag{3.36}$$

where each of the inclusions is open, and each of the vertical homomorphisms is perfect.

For each $k \geq n$ let $\pi_{n,k}: G_{n,\infty} \rightarrow G_{n,k}$ denote the universal homomorphism for the inverse limit $G_{n,\infty}$ (which is perfect by Corollary 3.3.3). Then $(\pi_{n,n}, G_{n,\infty})$ is a partial morphism from G_∞ to G_n . It follows from the universal properties of the inverse limits $G_{n,\infty}$, together with the universal property of the direct limit G_∞ , that G_∞ is an inverse limit in the category of second-countable locally compact étale groupoids with partial morphisms (see [AM18, Theorem 3.16]). The unit space of G_∞ can be identified with $E^{\leq \infty}$. \triangleleft

We finish this section by showing that the inverse limit G_∞ recovers \mathcal{TR}_E . First we require a more concrete description of the groupoids $G_{n,k}$ and $G_{n,\infty}$ and their topologies. To this end, for each $k > n$ consider the subgroupoid

$$H_{n,k} := \{(x, y) \in R_k \mid x_{[n+1,k]} = y_{[n+1,k]}\}$$

of R_k . In particular, $H_k = H_{k,k+1}$. For notational convenience we also let $H_{n,n} = R^n$. The arguments of Lemma 3.3.20 show that $H_{n,k}$ is clopen in R_k . Given the definition $G_{n,k}$ we can repeatedly apply (3.31) to see that as sets,

$$G_{n,k} = G_n \cup (\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n) = \left(\bigsqcup_{i=0}^n R_i \right) \sqcup \left(\bigsqcup_{i=n+1}^k H_{n,i} \right)$$

for all $k > n$, and $(\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n) = \bigsqcup_{i=n}^k H_{n,i} = R_n \sqcup \bigsqcup_{i=n+1}^k H_{n,i}$. Now define,

$$H_{n,\infty} := \{(x, y) \in E^\infty \times E^\infty \mid \sigma^n(x) = \sigma^n(y)\}.$$

Since $G_{n,\infty}$ is defined as an inverse limit we can make the identification

$$\begin{aligned} G_{n,\infty} &= G_n \sqcup \pi_{n,k}^{-1}(R_n) = G_n \sqcup \left(\bigsqcup_{i=n+1}^{\infty} H_{n,i} \right) \sqcup H_{n,\infty} \\ &= \{(x, y) \in E^{\leq \infty} \times E^{\leq \infty} \mid \exists i \leq n \text{ such that } x_{[i+1,\infty]} = y_{[i+1,\infty]}\}. \end{aligned} \quad (3.37)$$

Under this identification $\pi_{n,k}: G_{n,\infty} \rightarrow G_{n,k}$ is given by,

$$\pi_{n,k}(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in G_{n,k}. \\ (x_{[0,k]}, y_{[0,k]}) & \text{if } (x, y) \in H_{n,\infty} \sqcup \left(\bigsqcup_{i=k+1}^{\infty} H_{n,i} \right). \end{cases} \quad (3.38)$$

For each k such that $n \leq k < \infty$, $H_{n,k}$ is open in R_k which is in turn open in G_k .

Before we proceed, we give a more refined basis for \mathcal{TR}_E than the cylinder sets given in (3.3).

Lemma 3.3.30. *The groupoid \mathcal{TR}_E has a basis consisting of open sets of the form $\mathcal{Z}(\tilde{\rho}_{\infty,k}^{-1}(U), n, n, \tilde{\rho}_{\infty,k}^{-1}(V))$, where $n \in \mathbb{N}_0$, $k \geq n$, and $U, V \subseteq E^{\leq k}$ are open subsets of $E^{\leq k} \setminus E^{\leq n-1}$.*

Proof. Recall from (3.3) that \mathcal{TR}_E admits a basis of open sets of the form $\mathcal{Z}(U, n, n, V)$, where $n \in \mathbb{N}_0$, $k \geq n$, and $U, V \subseteq E^{\leq \infty} \setminus E^{n-1} = \text{dom}(\sigma^n)$. Recall from (3.24) that $E^{\leq \infty} \setminus E^{n-1} = \tilde{\rho}_{\infty,n}^{-1}(E^n)$. Since $E^{\leq \infty}$ is equipped with an inverse limit topology (see Section 3.3.1.1), we can assume that U and V are of the form $U = \tilde{p}_{\infty,k}^{-1}(U')$ and $V = \tilde{\rho}_{\infty,l}^{-1}(V')$ for $U' \subseteq E^{\leq k}$ and $V' \subseteq E^{\leq l}$ open. Without loss of generality we can assume that $l = k$, for if $l \geq k$, then $\tilde{p}_{\infty,k}^{-1}(U') = \tilde{p}_{\infty,l}^{-1}(\tilde{p}_{l,k}^{-1}(U'))$. Since $\tilde{\rho}_{\infty,k}^{-1}(U') \subseteq \tilde{\rho}_{\infty,n}^{-1}(E^n)$, it follows that

$$\tilde{\rho}_{\infty,k}^{-1}(U') = \tilde{\rho}_{\infty,k}^{-1}(U') \cap \tilde{\rho}_{\infty,n}^{-1}(E^n) = \begin{cases} \tilde{\rho}_{\infty,k}^{-1}(U' \cap \tilde{\rho}_{[k,n]}^{-1}(E^n)) & \text{if } k \geq n, \\ \tilde{\rho}_{\infty,n}^{-1}(\tilde{\rho}_{[n,k]}^{-1}(U') \cap E^n) & \text{if } n > k. \end{cases}$$

Considering E^n as an open subset of $E^{\leq n}$, we have $E^n = E^{\leq n} \setminus E^{\leq n-1}$. Thus, if $n > k$ we have $\tilde{\rho}_{[n,k]}^{-1}(U') \cap E^n \subseteq E^{\leq n} \setminus E^{\leq n-1}$. On the other hand, if $k \geq n$, we use that $\tilde{\rho}_{[k,n]}: E^{\leq k} \rightarrow E^{\leq n}$ is surjective to see that $\tilde{\rho}_{[k,n]}^{-1}(E^n) = E^{\leq k} \setminus E^{\leq n-1}$. Hence, $U' \cap \tilde{\rho}_{[k,n]}^{-1}(E^n) \subseteq E^{\leq k} \setminus E^{\leq n-1}$ when $k \geq n$. Similar considerations for V' yield the result. \square

Consider the open subgroupoid,

$$\begin{aligned} \mathcal{TR}_E^n &:= \bigcup_{i=0}^n \mathcal{Z}(E^{\leq \infty} \setminus E^{\leq i-1}, i, i, E^{\leq \infty} \setminus E^{\leq i-1}) \\ &= \{(x, y) \in \mathcal{TR}_E \mid \exists i \leq n \text{ such that } x_{[i+1,\infty]} = y_{[i+1,\infty]}\} \end{aligned}$$

of \mathcal{TR}_E , and note that $\mathcal{TR}_E = \bigcup_{n=0}^{\infty} \mathcal{TR}_E^n$. Comparing to both (3.37) and (3.30), we see that as sets \mathcal{TR}_E^n may be identified with $G_{n,\infty}$, and \mathcal{TR}_E may be identified with G_{∞} . The problem now falls to showing that the topology on G_{∞} agrees with the topology on \mathcal{TR}_E . Lemma 3.3.30 implies that

$$\{\mathcal{Z}(\tilde{\rho}_{\infty,k}^{-1}(U), n, n, \tilde{\rho}_{\infty,k}^{-1}(V)) \mid n \in \mathbb{N}_0, k \geq n, U, V \subseteq E^{\leq k} \setminus E^{\leq n-1} \text{ are open}\}$$

is a basis for the topology on \mathcal{TR}_E . Since each $G_{n,\infty}$ is open in G_∞ and is equipped with the initial topology induced by the maps $\pi_{n,k}$, the set

$$\{\pi_{k,n}^{-1}(U) \mid n \in \mathbb{N}, k \geq n, U \text{ open in } G_{n,k}\}$$

forms a basis for the topology on G_∞ .

Lemma 3.3.31. *Every open set in \mathcal{TR}_E is open in G_∞ .*

Proof. Fix a basic open set $\mathcal{Z}(\tilde{\rho}_{\infty,k}^{-1}(U), n, n, \tilde{\rho}_{\infty,k}^{-1}(V))$ for \mathcal{TR}_E with $k \geq n$ and U, V open in $E^{\leq k} \setminus E^{\leq n-1}$. Since the unit space of G_k is homeomorphic to $E^{\leq k}$, both $r_{G_k}^{-1}(U)$ and $s_{G_k}^{-1}(V)$ are open subsets of G_k , and indeed $G_k \setminus G_{n-1}$. Let $W = r_{G_k}^{-1}(U) \cap s_{G_k}^{-1}(V)$. Since $H_{n,k}$ is an open subgroupoid of G_k ,

$$W \cap H_{n,k} = \{(x, y) \in G_k \mid x \in U, y \in V, x_{[n+1,k]} = y_{[n+1,k]}\}$$

is open in G_k . It now follows that,

$$\begin{aligned} \pi_{k,k}^{-1}(W \cap H_{n,k}) &= \{(x, y) \in \mathcal{TR}_E \mid x_{[0,k]} \in U, y_{[0,k]} \in V, x_{[n+1,\infty]} = y_{[n+1,\infty]}\} \\ &= \mathcal{Z}(\tilde{\rho}_{\infty,k}^{-1}(U), n, n, \tilde{\rho}_{\infty,k}^{-1}(V)). \end{aligned}$$

So, $\mathcal{Z}(\tilde{\rho}_{\infty,k}^{-1}(U), n, n, \tilde{\rho}_{\infty,k}^{-1}(V))$ is open in the initial topology on $G_{k,\infty}$, which is in turn open in G_∞ . Hence, every open set in \mathcal{TR}_E is open in G_∞ . \square

Before we proceed show that the topology on \mathcal{TR}_E is finer than the topology on G_∞ , we need to know more about the topology on G_∞ . To this end, we give a tractable description of the topology on the open subgroupoid $(\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n)$ of $G_{n,k}$ for $k > n$. The following technical lemma allows us to identify $(\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n)$ as a transformation groupoid, thereby giving us better handle on its topology.

Lemma 3.3.32. *Suppose that $n \in \mathbb{N}_0$, and consider R_n as an open subgroupoid of G_n . For each $k \geq n$ the open subset $E^{\leq k} \setminus E^{\leq n-1}$ of $E^{\leq k}$ is an R_n -space with moment map $m_k: E^{\leq k} \setminus E^{\leq n-1} \rightarrow R_n^{(0)}$ given by $m_k = \tilde{\rho}_n \circ \cdots \circ \tilde{\rho}_{k-1}$, and action given by*

$$(x, y) \cdot z = xz_{[n+1,\ell(z)]},$$

which is defined whenever $y = z_{[0,n]}$. For each $k \geq n$ the transformation groupoid $R_n \ltimes (E^{\leq k} \setminus E^{\leq n-1})$ is isomorphic—as a topological groupoid—to the open subgroupoid $(\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n) = \bigsqcup_{i=n}^k H_{n,i}$ of $G_{n,k}$. The isomorphism $\Sigma_{n,k}: R_n \ltimes (E^{\leq k} \setminus E^{\leq n-1}) \rightarrow (\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n)$ is given by

$$\Sigma_{n,k}((x, y), z) = (xz_{[n+1,\ell(z)]}, yz_{[n+1,\ell(z)]}) \quad (3.39)$$

with inverse

$$\Sigma_{n,k}^{-1}(x, y) = ((x_{[0,n]}, y_{[0,n]}), y).$$

Proof. We keep the argument brief, as we have already seen analogous arguments previous proofs. Since the moment map $m_k = \tilde{\rho}_n \circ \cdots \circ \tilde{\rho}_{k-1}$ is continuous, an argument similar to Lemma 3.2.23 implies that the action is continuous. Hence, $E^{\leq k} \setminus E^{\leq n-1}$ is an R_n -space.

For the isomorphism, note that when $k = n$ this is just the isomorphism between $R_n \ltimes R_n^{(0)}$ and R_n . Now suppose for induction that we have an isomorphism $\Sigma_{n,k}$ satisfying equation (3.39) and that the unit space of $(\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n)$ can be identified with $E^{\leq k} \setminus E^{\leq n-1}$, thought of as a subset of the unit space $G_k^{(0)} = E^{\leq k}$.

Recall from the construction of G_{k+1} that we considered the map $\rho_k: E^{k+1} \rightarrow R_k^{(0)} \subseteq G_k^{(0)}$, and the induced perfect map $\tilde{\rho}_k: E^{\leq k+1} \rightarrow R_k^{(0)} \subseteq G_k^{(0)}$. The G_k -equivariance of $\tilde{\rho}_k$ then induced a perfect groupoid homomorphism $\tilde{\rho}_k^\times: G_k \ltimes E^{\leq k+1} \rightarrow G_k$. Since $(\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n)$ is an open subgroupoid of G_k , it follows that

$$(\tilde{\rho}_{[k+1,n]}^\times)^{-1}(R_n) = (\tilde{\rho}_k^\times)^{-1} \circ (\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n) = (\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n) \ltimes E^{\leq k+1}.$$

By the inductive hypothesis, the unit space of $(\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n)$ is equal to $E^{\leq k} \setminus E^{\leq n-1}$. Consequently,

$$(\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n) \ltimes E^{\leq k+1} = (\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n) \ltimes (E^{\leq k+1} \setminus E^{n-1}).$$

We now use the isomorphism $\Sigma_{n,k}$ to see that

$$(\tilde{\rho}_{[k+1,n]}^\times)^{-1}(R_n) = \Sigma_{n,k}(R_n \ltimes (E^{\leq k} \setminus E^{\leq n-1})) \ltimes (E^{\leq k+1} \setminus E^{n-1}). \quad (3.40)$$

Recall that $R_n \ltimes (E^{\leq k} \setminus E^{\leq n-1})$ has unit space homeomorphic to $E^{\leq k} \setminus E^{\leq n-1}$. Remembering that E^k is open in $E^{\leq k}$ we can consider the continuous map $q_k: E^{k+1} \rightarrow E^k \hookrightarrow (R_n \ltimes (E^{\leq k} \setminus E^{\leq n-1}))^{(0)}$, which is identical to ρ_k , but has codomain $(R_n \ltimes (E^{\leq k} \setminus E^{\leq n-1}))^{(0)}$. We note that E^{k+1} is an $R_n \ltimes (E^{\leq k} \setminus E^{\leq n-1})$ -space with moment map q_k and action given by

$$((x, y), z) \cdot w = xw_{[n+1, k+1]},$$

which is defined when $z_{[0, n]} = y$, $z \in E^k$, and $w_{[0, k]} = z$. Again, an argument similar to Lemma 3.2.23 implies that the action is continuous.

Proposition C.3.17 shows that the action of $R_n \ltimes (E^{\leq k} \setminus E^{\leq n-1})$ on E^{k+1} can be lifted to an action of $R_n \ltimes (E^{\leq k} \setminus E^{\leq n-1})$ on $E^{\leq k+1} \setminus E^{\leq n-1}$ with moment map \tilde{q}_k and action satisfying

$$((x, y), z) \cdot w = \begin{cases} xz_{[n+1, k]} & \text{if } w = z \in E^{\leq k} \setminus E^{\leq n-1} \text{ and } z_{[0, n]} = y, \\ xw_{[n+1, k+1]} & \text{if } w \in E^{k+1}, z_{[0, n]} = y, z \in E^k, \text{ and } w_{[0, k]} = z. \end{cases} \quad (3.41)$$

It follows that $\Sigma_{n,k} \circ \tilde{q}_k = \tilde{\rho}_k$ and $\gamma \cdot w = \Sigma_{n,k}(\gamma) \cdot w$ for all $\gamma \in R_n \ltimes (E^{\leq k} \setminus E^{\leq n-1})$ and $w \in E^{\leq k+1} \setminus E^{n-1}$. Since $\Sigma_{n,k}$ is an isomorphism, it follows from (3.40) that there is a groupoid isomorphism $\alpha_{n,k+1}: (R_n \ltimes (E^{\leq k} \setminus E^{\leq n-1})) \ltimes (E^{\leq k+1} \setminus E^{n-1}) \rightarrow (\tilde{\rho}_{[k+1,n]}^\times)^{-1}(R_n)$ satisfying

$$\alpha_{n,k+1}(\gamma, x) = (\Sigma_{n,k}(\gamma), x)$$

for all $(\gamma, x) \in (R_n \ltimes (E^{\leq k} \setminus E^{\leq n-1})) \ltimes (E^{\leq k+1} \setminus E^{n-1})$.

Define $\beta_{n,k+1}: (R_n \ltimes (E^{\leq k} \setminus E^{\leq n-1})) \ltimes (E^{\leq k+1} \setminus E^{n-1}) \rightarrow R_n \ltimes (E^{\leq k+1} \setminus E^{\leq n-1})$ by

$$\beta_{n,k+1}(((x, y), z), w) = ((x, y), w).$$

Then $\beta_{n,k+1}$ is a topological groupoid isomorphism with inverse

$$\beta_{n,k+1}^{-1}((x, y), w) = \begin{cases} (((x, y), w), w) & \text{if } w \in E^{\leq k} \setminus E^{\leq n-1}, \\ (((x, y), w_{[0,k]}), w) & \text{if } w \in E^k, \end{cases}$$

which is well-defined because of the definition of the action (3.41).

It now follows that $\Sigma_{n,k+1} := \alpha_{n,k+1} \circ \beta_{n,k+1}^{-1}$ is a topological groupoid isomorphism from $R_n \ltimes (E^{\leq k+1} \setminus E^{\leq n-1})$ to $(\tilde{\rho}_{[k+1,n]}^\times)^{-1}(R_n)$, and one checks that $\Sigma_{n,k+1}$ satisfies (3.39). The unit space of $(\tilde{\rho}_{[k+1,n]}^\times)^{-1}(R_n)$ can be identified with the open subset $E^{\leq k+1} \setminus E^{\leq n-1}$ of $G_{k+1}^{(0)} = E^{\leq k+1}$. \square

The advantage of describing $(\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n)$ as a transformation groupoid is transformation groupoids have a tractable basis. Indeed, the collection

$$\mathcal{B}'_{n,k} := \{((U \times V) \cap R_n) \times_{s_{R_n}, m_k} W \mid U, V \subseteq E^n \text{ open}, W \subseteq E^{\leq k} \setminus E^{\leq n-1} \text{ open}\}$$

forms a basis for the topology on $R_n \ltimes (E^{\leq k} \setminus E^{\leq n-1})$. We can use the isomorphism $\Sigma_{n,k}$ of Lemma 3.3.32 to transfer this to a basis for $(\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n)$. To this end, note that under the identification of $(\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n)$ with $\bigsqcup_{i=n}^k H_{n,i}$ we have

$$\begin{aligned} \Sigma_{n,k}(((U \times V) \cap R_n) \times_{s_{R_n}, m_k} W) \\ &= \left\{ (x, y) \in \bigsqcup_{i=n}^k H_{n,i} \mid (x_{[0,n]}, y_{[0,n]}) \in (U \times V) \cap R_n, \text{ and } x, y \in W \right\} \\ &= \left\{ (x, y) \in \bigsqcup_{i=n}^k H_{n,i} \mid x \in \tilde{\rho}_{k,n}^{-1}(U) \cap W, y \in \tilde{\rho}_{k,n}^{-1}(V) \cap W \right\} \end{aligned}$$

for all $((U \times V) \cap R_n) \times_{s_{R_n}, m_k} W \in \mathcal{B}'_{n,k}$. Since $\tilde{\rho}_{k,n}^{-1}(U) \cap W$ and $\tilde{\rho}_{k,n}^{-1}(V) \cap W$ are already open in $E^{\leq k} \setminus E^{\leq n-1}$, we deduce the following.

Lemma 3.3.33. *Let $k \geq n$. For each pair of open subsets U, V of $E^{\leq k} \setminus E^{\leq n-1}$ let*

$$\mathcal{W}(U, V) := \left\{ (x, y) \in \bigsqcup_{i=n}^k H_{n,i} \mid x \in U, y \in V \right\}.$$

Then $\mathcal{B}'_{n,k} = \{\mathcal{W}(U, V) \mid U, V \subseteq E^{\leq k} \setminus E^{\leq n-1} \text{ open}\}$ is a basis for the topology on $(\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n) = \bigsqcup_{i=n}^k H_{n,i} \subseteq G_k$.

Armed with a reasonable basis for $(\tilde{\rho}_{[k,n]}^\times)^{-1}(R_n)$ can now prove that the topology on \mathcal{TR}_E is finer than the topology on G_∞ .

Lemma 3.3.34. *Every open set in G_∞ is open in \mathcal{TR}_E .*

Proof. Fix $(x, y) \in G_\infty$. Let $n \in \mathbb{N}_0$ be the smallest number such that $x_{[n+1,\infty]} = y_{[n+1,\infty]}$. It follows that $(x, y) \in G_{n,\infty}$, $\ell(l) = \ell(y) \geq n$, and $(x_{[0,n]}, y_{[0,n]}) \in R_n$. Fix a basic open neighbourhood $\pi_{k,n}^{-1}(W) \subset G_{n,\infty}$ about (x, y) , where $k \geq n$ and W is open in $G_{n,k}$. Then $(x_{[0,k]}, y_{[0,k]}) \in W$ (note that we could have $\ell(x) = \ell(y) < k$).

Since $(x_{[0,n]}, y_{[0,n]}) \in R_n$, it follows that $(x_{[0,k]}, y_{[0,k]}) \in (\tilde{\rho}_{[n,k]}^\times)^{-1}(R_n)$. In particular, $(x_{[0,k]}, y_{[0,k]})$ is an element of the open subset $S := (\tilde{\rho}_{[n,k]}^\times)^{-1}(R_n) \cap W$ of $(\tilde{\rho}_{[n,k]}^\times)^{-1}(R_n) \subseteq G_{n,k}$. By Lemma 3.3.33 there exist open sets U, V in $E^{\leq k} \setminus E^{\leq n-1}$ such that $\mathcal{W}(U, V)$ is an open neighbourhood of $(x_{[0,k]}, y_{[0,k]})$ contained in S . Consequently, $\pi_{n,k}^{-1}(\mathcal{W}(U, V)) = \mathcal{Z}(U, n, n, V)$ is an open neighbourhood of (x, y) contained in $\pi_{n,k}^{-1}(W)$. Hence, every open set in G_∞ topology is open in \mathcal{TR}_E . \square

We finally prove that the inverse limit G_∞ agrees with \mathcal{TR}_E .

Theorem 3.3.35. *Let E be a topological graph. Then the groupoid G_∞ is isomorphic to \mathcal{TR}_E as a topological groupoid.*

Proof. This follows immediately from Lemma 3.3.31 and Lemma 3.3.34. \square

3.3.2.2 Reconstructing \mathcal{R}_E

We now move on to reconstructing the core \mathcal{R}_E of the boundary path groupoid for a topological graph E . To do so we construct an étale groupoid $G\mathcal{O}_k$, for each $k \in \mathbb{N}_0$, such that $C^*(G\mathcal{O}_k)$ is isomorphic to $C_{[0,k]}$. Then, by once again taking an inverse limit of the groupoids $G\mathcal{O}_k$ with respect to partial morphisms, we recover \mathcal{R}_E . The construction of $G\mathcal{O}_k$ is analogous to the construction of G_k from Section 3.3.2.1, where the unified space construction is replaced by the minimal perfection. Accordingly, we keep the description brief to save repetition, and direct the reader to Section 3.3.2.1 to fill in the details.

Construction: $G\mathcal{O}_k$

We start by setting $G\mathcal{O}_0 = R_0$. Taking the minimal perfection of E^1 with respect to $\rho_0: E^1 \rightarrow E^0$, we arrive at $E^{1,+} = E^1 \sqcup E_{sing}^0$ (see Proposition 3.3.14). Then ρ_0 admits a perfect extension $\rho_0^+: E^{1,+} \rightarrow E^0$. Since E^1 is open in $E^{1,+}$, we can form the adjunction groupoid,

$$G\mathcal{O}_1 := R_1 \sqcup_{\Sigma_1} E^{1,+},$$

where $\Sigma_1: R_1^{(0)} \cong E^0 \hookrightarrow E^{1,+}$ is the open inclusion. Theorem C.2.10 implies that $G\mathcal{O}_1$ is étale, and has unit space $G\mathcal{O}_1^{(0)} = E^{1,+}$. Since $E^{1,+}$ is a closed subspace of $E^{\leq 1}$, $G\mathcal{O}_1$ can be identified as a closed subgroupoid of G_1 .

To build $G\mathcal{O}_2$, we start with the continuous open map $\rho_1: E^2 \rightarrow E^1 \hookrightarrow E^{1,+}$. The canonical action of $G\mathcal{O}_1$ on its unit space $E^{1,+}$ lifts, via ρ_1 , to an action of $G\mathcal{O}_1$ on E^2 . The corresponding transformation groupoid $G\mathcal{O}_1 \ltimes E^2$ can again be identified with $R_1 \ltimes E^2$, because elements of $G\mathcal{O}_1$ which belong E_{sing}^0 play no role in the action of $G\mathcal{O}_1$ on E^2 . Applying Corollary C.3.18 we see that the minimal perfection $E^{2,+}$ is a $G\mathcal{O}_1$ -space, and $\rho_1^+: E^{2,+} \rightarrow E^{1,+}$ is $G\mathcal{O}_1$ -equivariant. Lemma C.3.20 now implies that there is an induced perfect map $\rho_1^{+\times}: G\mathcal{O}_1 \ltimes E^{2,+} \rightarrow G\mathcal{O}_1$ satisfying,

$$\rho_1^{+\times}(\gamma, z) = \gamma.$$

We can then identify $G\mathcal{O}_1 \ltimes E^{2,+}$ with $(R_1 \ltimes E^2) \sqcup R_1^{sing} \sqcup R_0^{sing}$, where $R_1 \ltimes E^2$ is identified with the restriction of $G\mathcal{O}_1 \ltimes E^{2,+}$ to $E^2 \subseteq E^{2,+}$.

Since H_2 from (3.29) is a clopen subgroupoid of R_2 , and Lemma 3.3.21 gives an isomorphism $\Sigma_2: H_2 \rightarrow R_1 \ltimes E^2$. We then apply Theorem C.2.10 to construct the adjunction space,

$$G\mathcal{O}_2 := R_2 \sqcup_{\Sigma_2} G\mathcal{O}_1 \ltimes E^{2,+},$$

which has the structure of an étale groupoid. The groupoid $G\mathcal{O}_2$ has unit space $E^{2,+}$. The set $E^2 \subseteq G\mathcal{O}_2^{(0)}$ is open and $G\mathcal{O}_2$ -invariant and the restriction $G\mathcal{O}_2|_{E^2}$ can be identified with R_2 . Again $G\mathcal{O}_2$ can be viewed a closed subgroupoid of G_2 . Iterating the process above we can similarly construct $G\mathcal{O}_k$ \triangleleft

To summarise we record the following.

Proposition 3.3.36. *Let E be a topological graph. For each $k \in \mathbb{N}$ there are amenable étale groupoids defined by setting $G\mathcal{O}_0 = R_0$ and inductively defining,*

$$G\mathcal{O}_k := R_k \sqcup_{\Sigma_k} (G\mathcal{O}_{k-1} \rtimes E^{k,+}),$$

where $\Sigma_k: H_k \rightarrow R_{k-1} \rtimes E^k$ is the isomorphism of Lemma 3.3.21. Moreover, $G\mathcal{O}_k$ has the following properties:

- (i) the unit space $G\mathcal{O}_k$ is homeomorphic to $E^{k,+}$;
- (ii) E^k is an open $G\mathcal{O}_k$ -invariant subset of $G\mathcal{O}_k^{(0)}$, and $G\mathcal{O}_k|_{E^k}$ can be identified with R_k as a topological groupoid;
- (iii) $E^{k,+} \setminus E^k = \bigsqcup_{i=0}^{k-1} E_{sing}^i$ is a closed $G\mathcal{O}_k$ -invariant subset of $G\mathcal{O}_k^{(0)}$, and $G\mathcal{O}_k|_{\bigsqcup_{i=0}^{k-1} E_{sing}^i}$ can be identified with $\bigsqcup_{i=0}^{k-1} R_i^{sing}$ as a topological groupoid;
- (iv) as sets we have a decomposition $G\mathcal{O}_k = R_k \sqcup \bigsqcup_{i=0}^{k-1} R_i^{sing}$, so that

$$G\mathcal{O}_k = \{(x, y) \in E^{k,+} \times E^{k,+} \mid \ell(x) = \ell(y) \text{ and } s(x) = s(y)\};$$

- (v) $G\mathcal{O}_k \rtimes E^{k+1,+}$ can be identified with an open subgroupoid of $G\mathcal{O}_{k+1}$, and as sets

$$G\mathcal{O}_k \rtimes E^{k+1,+} = H_{k+1} \sqcup \bigsqcup_{i=0}^k R_i^{sing};$$

- (vi) there is a perfect groupoid homomorphism $\rho_k^{+\times}: G\mathcal{O}_k \rtimes E^{k+1,+} \rightarrow G\mathcal{O}_k$ satisfying $\rho_k^{+\times}(\gamma, x) = \gamma$, which upon identifying $G\mathcal{O}_k \rtimes E^{k+1,+}$ with $H_{k+1} \sqcup \bigsqcup_{i=0}^k R_i^{sing}$ satisfies,

$$\rho_k^{+\times}(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in \bigsqcup_{i=0}^k R_i^{sing} \\ (x_{[0,k]}, y_{[0,k]}) & \text{if } (x, y) \in H_{k+1}; \text{ and} \end{cases} \quad (3.42)$$

- (vii) $G\mathcal{O}_k$ can be identified with a closed subgroupoid of G_k .

Remark 3.3.37. In the case where $E^0 = E^{reg}$ —that is when A acts by compact operators on the module X_E —the map $\rho_k: E^{k+1} \rightarrow E^k$ is already perfect and so the minimal perfections $E^{k,+}$ are just E^k for each $k \in \mathbb{N}_0$. Correspondingly, the groupoids $G\mathcal{O}_k$ are R_k for each $k \in \mathbb{N}_0$. This is the case in [DM01].

Remark 3.3.38. Just as in the construction of G_k , we have used both the adjunction groupoid construction and the fact that H_k is open in R_k , to construct $G\mathcal{O}_k$. Accordingly, the construction of $G\mathcal{O}_k$ does not apply to general topological quivers.

To show that $C^*(G\mathcal{O}_k)$ is isomorphic to $C_{[0,k]}$ we once again employ Proposition 3.2.5. Let $\alpha_k^+ : C^*(G\mathcal{O}_k) \rightarrow C^*(G\mathcal{O}_k \ltimes E^{k+1,+})$ denote the $*$ -homomorphism induced by the perfect homomorphism $\rho_k^{+\times}$, and let $\beta_k^+ : C^*(G\mathcal{O}_k \ltimes E^{k+1,+}) \rightarrow C^*(G\mathcal{O}_{k+1})$ denote the $*$ -homomorphism induced by the open inclusion of groupoids. Let $\tau_k^+ = \beta_k^+ \circ \alpha_k^+$. The following result is analogous to Corollary 3.3.24.

Proposition 3.3.39. *Let E be a topological graph. Fix $k \in \mathbb{N}_0$, and let τ_k be the $*$ -homomorphism induced by (3.23). Then the diagram*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C^*(R_k^{reg}) & \longrightarrow & C^*(G\mathcal{O}_k) & \longrightarrow & C^*(G\mathcal{O}_k \setminus R_k^{reg}) \longrightarrow 0 \\
 & & \downarrow \tau_k & & \downarrow \tau_k^+ & & \downarrow \cong \\
 0 & \longrightarrow & C^*(R_{k+1}) & \longrightarrow & C^*(G\mathcal{O}_{k+1}) & \longrightarrow & C^*(G\mathcal{O}_{k+1} \setminus R_{k+1}) \longrightarrow 0
 \end{array} \quad (3.43)$$

commutes and has exact rows.

Proof. Recall S_{k+1} , α_k , and σ_{k+1} from Section 3.2.3. Our aim is to show that the following diagram commutes and has exact rows:

$$\begin{array}{ccccccc}
 0 & \rightarrow & C^*(R_k^{reg}) & \longrightarrow & C^*(G\mathcal{O}_k) & \longrightarrow & C^*(G\mathcal{O}_k \setminus R_k^{reg}) \longrightarrow 0 \\
 & & \downarrow \sigma_{k+1} \circ \alpha_k & & \downarrow \alpha_k^+ & & \downarrow \cong \\
 & & C^*(S_{k+1}) & & & & \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & C^*(H_{k+1}) & \rightarrow & C^*(G\mathcal{O}_k \ltimes E^{k+1,+}) & \rightarrow & C^*((G\mathcal{O}_k \ltimes E^{k+1,+}) \setminus H_{k+1}) \rightarrow 0 \\
 & & \downarrow & & \downarrow \beta_k^+ & & \downarrow \cong \\
 0 & \rightarrow & C^*(R_{k+1}) & \longrightarrow & C^*(G\mathcal{O}_{k+1}) & \longrightarrow & C^*(G\mathcal{O}_{k+1} \setminus R_{k+1}) \longrightarrow 0
 \end{array} \quad (3.44)$$

That the diagram consisting of the bottom two rows of (3.44), together with the vertical $*$ -homomorphisms between them commutes and has exact rows, follows immediately from Corollary C.2.15.

Since $(R_k^{reg})^{(0)} = E_k^{reg}$ is an open R_k -invariant subset of $R_k^{(0)} = E^k$, and E^k is an open $G\mathcal{O}_k$ -invariant subset of $G\mathcal{O}_k^{(0)} = E^{k,+}$, it follows that $(R_k^{reg})^{(0)}$ is an open $G\mathcal{O}_k$ -invariant subset of $G\mathcal{O}_k$. Moreover, the restriction of $G\mathcal{O}_k$ to E_k^{reg} can be identified with R_k^{reg} . Theorem B.2.9 then gives the exact sequence along the top row of (3.44). We can also identify $G\mathcal{O}_k \setminus R_k^{reg}$ with $\bigsqcup_{i=0}^k R_i^{sing}$, which is closed in $G\mathcal{O}_k$.

Considering the left-most column of (3.44) we see that S_{k+1} and H_{k+1} are both open subgroupoids of R_{k+1} (see Lemma 3.2.24 and Lemma 3.3.20). Consequently, S_{k+1} is an open subgroupoid of H_{k+1} , and this induces an injective $*$ -homomorphism on the level of C^* -algebras. Moreover, the composition of $*$ -homomorphisms down the left column agrees with τ_k .

Commutativity of the top left square of (3.44) follows from comparing the formula for $\sigma_{k+1} \circ \alpha_k$ (see Lemma 3.2.25 and Equation (3.18)) to the formula (3.42), which are both defined on $C_c(R_k^{reg})$. Since $C_c(R_k^{reg})$ is dense in $C^*(R_k^{reg})$ the top left square commutes.

Finally, Proposition 3.3.36 implies that $(G\mathcal{O}_k \ltimes E^{k+1,+}) \setminus H_{k+1}$ can be identified with $\bigsqcup_{i=0}^k R_i^{sing}$, and the topologies on $G\mathcal{O}_k \setminus R_k^{reg}$ and $(G\mathcal{O}_k \ltimes E^{k+1,+}) \setminus H_{k+1}$ agree. The

restriction of $\rho_k^{+\times}$ to $(G\mathcal{O}_k \times E^{k+1,+}) \setminus H_{k+1}$ is the identity map on $\bigsqcup_{i=0}^k R_i^{sing}$ giving commutativity of the top right square in (3.44). \square

For $0 \leq l < k$ define $\tau_{[l,k]}^+ := \tau_k^+ \circ \dots \circ \tau_l^+ : C^*(G_l) \rightarrow C^*(G_k)$. Also, let $\tau_{[k,k]}^+ = \text{id}_{C^*(G_k)}$. Similar arguments to the proof of Lemma 3.3.26 yield the following.

Lemma 3.3.40. *Let E be a topological graph. $k \in \mathbb{N}_0$ and $m \leq k$. Recall the multiplication $*_{k,m} : C^*(R_k) \times C^*(R_m) \rightarrow C^*(R_k)$ of Proposition 3.2.20. Then for all $a \in C^*(R_k)$ and $b \in C^*(R_m)$ we have*

$$a\tau_{[m,k]}^+(b) = a *_{k,m} b \quad \text{and} \quad \tau_{[m,k]}^+(b)a = b *_{m,k} a.$$

We are now once again in the position to apply Proposition 3.2.5 to see that $C^*(G\mathcal{O}_k)$ is isomorphic to $C_{[0,k]}$.

Theorem 3.3.41. *Let E be a topological graph. For each $\ell \in \mathbb{N}_0$ let $\Phi_\ell : C^*(R_\ell) \rightarrow \text{End}_A^0(X_E^{\otimes \ell})$ be the isomorphism of Corollary 3.2.17. For each $k \in \mathbb{N}$ we have an isomorphism $\Upsilon_k^+ : C^*(G\mathcal{O}_k) \rightarrow C_{[0,k]}$, where $\Upsilon_0^+ = i_A$, and for all $k \in \mathbb{N}_0$ and all $l \leq k$ the diagram*

$$\begin{array}{ccc} C^*(R_l) & \xrightarrow{\Phi_l} & \text{End}_A^0(X_E^{\otimes l}) \\ \tau_{[l,k]}^+ \downarrow & & \downarrow i_{X_E}^{(l)} \\ C^*(G\mathcal{O}_k) & \xrightarrow{\Upsilon_k^+} & C_{[0,k]} \end{array} \quad (3.45)$$

commutes.

Proof. It follows from Lemma 3.3.40 and Proposition 3.2.20 that $(C^*(G\mathcal{O}_k), \tau_k^+)_{k \in \mathbb{N}_0}$ satisfies condition (iii) of Proposition 3.2.5. Condition (ii) of Proposition 3.2.5 follows from Proposition 3.3.39, and condition (i) is satisfied since $C^*(R_0) \cong C_0(E^0)$. The result now follows from Proposition 3.2.5. \square

Proposition 3.2.1 yields the following.

Corollary 3.3.42. *We have an isomorphism $\varprojlim (C^*(G\mathcal{O}_k), \tau_k^+) \cong \mathcal{O}_{X_E}^{\mathbb{T}}$*

Finally, to reconstruct \mathcal{R}_E , notice that $(\widetilde{\rho}_k^{\times}, G\mathcal{O}_k \times E^{k+1,+})_{k \in \mathbb{N}_0}$ defines a partial morphism from $G\mathcal{O}_{k+1}$ to $G\mathcal{O}_k$. Hence, we can define the inverse limit,

$$G\mathcal{O}_\infty := \varprojlim (\widetilde{\rho}_k^{\times}, G\mathcal{O}_k \times E^{k+1,+}).$$

Finally, $G\mathcal{O}_\infty$ agrees with the groupoid \mathcal{R}_E .

Theorem 3.3.43. *Let E be a topological graph. Then $G\mathcal{O}_\infty$ is isomorphic to \mathcal{R}_E as a topological groupoid.*

Proof. Since $G\mathcal{O}_k$ can be identified with a closed subgroupoid of G_k for all $k \in \mathbb{N}_0$, it follows from an analogous diagram to (3.36) that $G\mathcal{O}_\infty$ can be identified with a closed subgroupoid of G_∞ . In a similar manner to how G_∞ was identified as a set with \mathcal{TR}_∞ , $G\mathcal{O}_\infty$ can be identified as a set with \mathcal{R}_E . Since \mathcal{R}_E is a closed subgroupoid of \mathcal{TR}_E , it follows that the isomorphism of Theorem 3.3.35 restricts to an isomorphism between $G\mathcal{O}_\infty$ and \mathcal{R}_E . \square

CHAPTER 4

Detecting the Critical Set

In this chapter we always assume that (\mathbb{A}, Γ) is an injective iterated function system. Recall from Definition 1.2.12 that the *critical set* of (\mathbb{A}, Γ) is the subset

$$C_\Gamma = \bigcup_{\gamma \neq \gamma' \in \Gamma} \gamma(\mathbb{A}) \cap \gamma'(\mathbb{A}),$$

of \mathbb{A} , while the *post-critical set* of (\mathbb{A}, Γ) is

$$P_\Gamma := \bigcup_{w \in \mathbb{F}_N^+ \setminus \{\emptyset\}} \gamma_w^{-1}(C_\Gamma).$$

As we observed in Chapter 2, neither the algebra \mathcal{O}_X nor $C^*(\mathbb{A}, \Gamma)$ successfully detect information pertaining to the critical or post-critical sets of (\mathbb{A}, Γ) . Although $C^*(\mathbb{A}, \Gamma)$ does detect the image of the branched set $p_1(B_\Gamma)$ —which is a subset of C_Γ —if $C^*(\mathbb{A}, \Gamma)$ has no branched points and admits a code map, then $C^*(\mathbb{A}, \Gamma)$ is isomorphic to the Cuntz algebra $\mathcal{O}_{|\Gamma|}$ by Proposition 2.3.2.

In this chapter we revisit the correspondence X_Γ from Section 2.1 and its relationship to the critical set and post-critical set. We define a new correspondence Y_Γ by modifying X_Γ , and associate to it a new C^* -algebra \mathcal{O}_Y , which we call the lacunary algebra. The lacunary algebra has a greater sensitivity to the critical set than either of previously considered C^* -algebras. In particular, if $C_\Gamma \neq \emptyset$ then \mathcal{O}_Y is typically not a Cuntz algebra.

To build \mathcal{O}_Y we introduce the *singular boundary* of an iterated function system. The singular boundary is a subset of the post-critical set that is defined for iterated function systems with fat overlap. Proposition 4.1.13 implies that the singular boundary is in some sense an obstruction to continuously inverting the dynamics of an iterated function system. We also introduce the notion of *post-critically stable* iterated function systems as a weakening of the post-critically finite condition. Typically, we work in the post-critically stable setting.

Once we have established some basic facts about \mathcal{O}_Y , the K -theory of \mathcal{O}_Y is computed for some illustrative examples. The K -theory computations further reveal that \mathcal{O}_Y is more sensitive to the interplay between the topology and dynamics of the underlying iterated function system than either \mathcal{O}_X or $C^*(\mathbb{A}, \Gamma)$.

We finish this chapter by exploring the relationship between critical points and inner products on modules. We show that if an iterated function system has a non-trivial critical boundary, then the C^* -correspondence X_Γ does not admit a natural left inner product. The results of this chapter are entirely new.

4.1 | The singular boundary

Let $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ be an injective iterated function system with code map $\pi: \Omega_N \rightarrow \mathbb{A}$. In this section we revisit the isomorphism $\Phi: \mathcal{O}_{X_\Gamma} \rightarrow \mathcal{O}_N$ from Proposition 2.1.4. For each $w \in \mathbb{F}_N^+$ let $\kappa_w: C(\mathbb{A}) \rightarrow C(\mathbb{A})$ be the $*$ -homomorphism dual to γ_w . That is,

$$\kappa_w(a)(x) = a(\gamma_w(x)). \quad (4.1)$$

On the code space $(\Omega_N, \bar{\Gamma})$ we denote by $\bar{\kappa}_w$ the $*$ -homomorphism dual to $\bar{\gamma}_w$. For $w, v \in \mathbb{F}_N^+$ we have

$$\begin{aligned} \bar{\kappa}_v(\chi_{\mathcal{Z}(w)})(u) &= \begin{cases} 1 & \text{if } vu \in \mathcal{Z}(w), \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \chi_{\mathcal{Z}(v^{-1}w)}(u) & \text{if } v \leq w, \\ 1 & \text{if } w \leq v, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (4.2)$$

Recall that $\mathbb{A}_w := \gamma_w(\mathbb{A})$. For each $a \in C(\mathbb{A})$ and $w \in \mathbb{F}_N^+$ consider the function $\tau_w(a)$ on \mathbb{A} defined by

$$\tau_w(a)(x) = \begin{cases} a(\gamma_w^{-1}(x)) & \text{if } x \in \mathbb{A}_w, \\ 0 & \text{otherwise.} \end{cases} \quad (4.3)$$

Typically, the function $\tau_w(a)$ is not continuous, but it is always bounded and Borel since \mathbb{A}_w is closed. Letting $\mathcal{B}(\mathbb{A})$ denotes the bounded Borel measurable functions on \mathbb{A} , $a \mapsto \tau_w(a)$ defines a $*$ -homomorphism $\tau_w: C(\mathbb{A}) \rightarrow \mathcal{B}(\mathbb{A})$. Injectivity of γ_w implies that $\tau_w \circ \kappa_w = 1_{C(\mathbb{A})}$, so τ_w is a one-sided inverse to κ_w . We can characterise precisely when $\tau_w(a)$ is continuous.

Lemma 4.1.1. *Let $a \in C(\mathbb{A})$. Then $\tau_w(a) \in C(\mathbb{A})$ if and only if $a \in C_0(\mathbb{A} \setminus \gamma_w^{-1}(\partial\mathbb{A}_w))$.*

Proof. We need some preliminary work for both implications. Injectivity of γ_w together with compactness of \mathbb{A} implies that γ_w defines a homeomorphism from \mathbb{A} to \mathbb{A}_w . In particular, γ_w^{-1} is a homeomorphism from $\text{int}(\mathbb{A}_w) = \mathbb{A}_w \setminus \partial\mathbb{A}_w$ to $\mathbb{A} \setminus \gamma_w^{-1}(\partial\mathbb{A}_w)$. Let $\varphi: C_0(\mathbb{A} \setminus \gamma_w^{-1}(\partial\mathbb{A}_w)) \rightarrow C_0(\text{int}(\mathbb{A}_w))$ denote the induced isomorphism, and let $\epsilon: C_0(\text{int}(\mathbb{A}_w)) \rightarrow C(\mathbb{A})$ denote the standard inclusion.

Now suppose that $a \in C_0(\mathbb{A} \setminus \gamma_w^{-1}(\partial\mathbb{A}_w))$. Then $\tau_w(a) = \epsilon(\varphi(a))$, so τ_w is continuous. For the reverse implication, we prove the contrapositive. Suppose that $a \in C(\mathbb{A})$ and $a(x) \neq 0$ for some $x \in \gamma_w^{-1}(\partial\mathbb{A}_w)$. Then $\tau_w(a)(\gamma_w(x)) = a(x) \neq 0$. Since $\gamma_w(x) \in \partial\mathbb{A}_w$, and $\tau(a)(y) = 0$ for all $y \in \mathbb{A} \setminus \mathbb{A}_w$, we see that $\tau(a)$ is not continuous. \square

Denote by $\bar{\tau}_w$ the map τ_w for the code space $(\Omega_N, \bar{\Gamma})$. Since $\bar{\gamma}_w(\Omega_N) = \mathcal{Z}(w)$ is clopen in Ω_N , it has empty boundary. Accordingly, Lemma 4.1.1 implies that $\bar{\tau}_w(a)$ is continuous

for all $a \in C(\Omega_N)$. For $w, v \in \mathbb{F}_N^+$ we have,

$$\begin{aligned} \bar{\tau}_v(\chi_{\mathcal{Z}(w)})(u) &= \begin{cases} \chi_{\mathcal{Z}(w)}(\bar{\gamma}_v^{-1}(u)) & \text{if } u \in \mathcal{Z}(v), \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1 & \text{if } v^{-1}u \in \mathcal{Z}(w), \\ 0 & \text{otherwise,} \end{cases} \\ &= \chi_{\mathcal{Z}(vw)}(u). \end{aligned} \tag{4.4}$$

With the functions $\bar{\kappa}_v$ and $\bar{\tau}_v$ we can see how the generating isometries of \mathcal{O}_N “act on” $C(\Omega_N)$.

Proposition 4.1.2. *Let $(\Omega_N, \bar{\Gamma})$ be the code space on N letters, and consider the Cuntz algebra \mathcal{O}_N . Let $\alpha: C(\Omega_N) \rightarrow \mathcal{D}_N$ be the canonical isomorphism satisfying $\alpha(\chi_{\mathcal{Z}(w)}) = S_w S_w^*$. Then for each $a \in C(\Omega_N)$ and $v \in \mathbb{F}_N^+$ we have*

$$S_v^* \alpha(a) S_v = \alpha(\bar{\kappa}_v(a)) \quad \text{and} \quad S_v \alpha(a) S_v^* = \alpha(\bar{\tau}_v(a)). \tag{4.5}$$

Proof. Let $w \in \mathbb{F}_N^+$. Using (4.2) we see that,

$$\begin{aligned} S_v^* \alpha(\chi_{\mathcal{Z}(w)}) S_v &= S_v^* S_w S_w^* S_v = \begin{cases} S_{v^{-1}w} S_{v^{-1}w}^* & \text{if } v \leq w, \\ 1 & \text{if } w \leq v, \\ 0 & \text{otherwise,} \end{cases} \\ &= \alpha(\bar{\kappa}_v(\chi_{\mathcal{Z}(w)})). \end{aligned}$$

On the other hand (4.4) gives

$$S_v \alpha(\chi_{\mathcal{Z}(w)}) S_v^* = S_v S_w S_w^* S_v^* = S_{vw} S_{vw}^* = \alpha(\chi_{\mathcal{Z}(vw)}) = \alpha(\bar{\tau}_v(\chi_{\mathcal{Z}(w)})).$$

Since the characteristic functions $\{\chi_{\mathcal{Z}(w)} \mid w \in \mathbb{F}_N^+\}$ span a dense subspace of $C(\Omega_N)$, the result now follows. \square

Recall that if (\mathbb{A}, Γ) has a code map $\pi: \Omega_N \rightarrow \mathbb{A}$, then $\pi \circ \bar{\gamma}_w = \gamma_w \circ \pi$ for all $w \in \mathbb{F}_N^+$, and $\mathbb{A}_w = \pi(\mathcal{Z}(w))$. For each $a \in C(\mathbb{A})$, it follows that $(\pi^* \circ \tau_w)(a) := \bar{\tau}_w(a \circ \pi)$ is continuous on Ω_N , even if $\tau_w(a)$ is not continuous on \mathbb{A} . Consequently, we have the following corollary to Proposition 4.1.2.

Corollary 4.1.3. *Let $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ be an injective iterated function system with code map $\pi: \Omega_N \rightarrow \mathbb{A}$. Let $\alpha: C(\Omega_N) \rightarrow \mathcal{D}_N$ be the canonical isomorphism satisfying $\alpha(\chi_{\mathcal{Z}(w)}) = S_w S_w^*$. Then for each $a \in C(\mathbb{A})$ and $v \in \mathbb{F}_N^+$,*

$$S_v^* (\alpha \circ \pi^*(a)) S_v = (\alpha \circ \pi^*)(\kappa_v(a)) \quad \text{and} \quad S_v (\alpha \circ \pi^*(a)) S_v^* = (\alpha \circ \pi^*)(\tau_v(a)).$$

For notational clarity we identify $C(\mathbb{A})$ with its image $(\alpha \circ \pi^*)(C(\mathbb{A}))$. With this

convention Corollary 4.1.3 states that

$$S_v^* a S_v = \kappa_v(a) \quad \text{and} \quad S_v a S_v^* = \tau_v(a), \quad (4.6)$$

for all $a \in C(\mathbb{A})$ and $v \in \mathbb{F}_N^+$. Observe that $S_v a S_v^*$ no longer corresponds to a continuous function on \mathbb{A} , but instead to a bounded Borel function $\tau_v(a)$. In this sense, the conjugation $a \mapsto S_v a S_v^*$ does not preserve the topology of the underlying attractor \mathbb{A} .

Recall from Proposition 2.1.4 that $\Phi \circ i_{C(\mathbb{A})}(a) = (\alpha \circ \pi^*)(a)$ for all $a \in C(\mathbb{A})$. The observation that the topology is not preserved somewhat explains why \mathcal{O}_{X_Γ} is always isomorphic to \mathcal{O}_N . Heuristically, the Cuntz-Pimsner construction applied to X_Γ forcibly “inverts the dynamics” of the iterated function system (\mathbb{A}, Γ) , without regard for the topology of \mathbb{A} . This process “disconnects” \mathbb{A} which is why we end up with the Cuntz algebra \mathcal{O}_N whose diagonal \mathcal{D}_N consists of functions on a totally-disconnected Cantor space Ω_N .

Remark 4.1.4. Considering $C(\mathbb{A})$ as a subalgebra of $\mathcal{D}_N \subseteq \mathcal{O}_N$ we see that $C(\mathbb{A})$ is a long way from being a Cartan subalgebra of \mathcal{O}_N in the sense of [Ren08, Definition 5.1]. Indeed, $C(\mathbb{A})$ is not maximal Abelian since it is contained in \mathcal{D}_N . Moreover, since $S_v C(\mathbb{A}) S_v^*$ is not a subset of $C(\mathbb{A})$ we find that $C(\mathbb{A})$ is not regular. There is also no obvious conditional expectation from \mathcal{O}_N to $C(\mathbb{A})$.

As an attempt to overcome the disconnectedness described by the algebra \mathcal{O}_{X_Γ} , we take a novel approach, removing the points of \mathbb{A} for which the functions $\tau_w(a)$ are discontinuous. We then—in Section 4.2—adapt the construction of X_Γ to build a C^* -correspondence over C_0 -functions on \mathbb{A} with the problematic points removed. Removing points may at first seem counter-intuitive, since the problems we have encountered are due to the topology on \mathbb{A} being disconnected. However, our passage from a compact space to a locally compact space retains a remnant of the topology of the points we have removed, at the point at infinity in the one-point compactification.

Definition 4.1.5. The *singular boundary* of an injective iterated function system (\mathbb{A}, Γ) is the set

$$\Delta P_\Gamma := \bigcup_{w \in \mathbb{F}_N^+} \gamma_w^{-1}(\partial \mathbb{A}_w). \quad (4.7)$$

As the notation might suggest, the singular boundary is always a subset of P_Γ .

Lemma 4.1.6. *Let (\mathbb{A}, Γ) be an injective iterated function system. Then $\Delta P_\Gamma \subseteq P_\Gamma$.*

Proof. Fix $x \in \Delta P_\Gamma$ and let $\Gamma = \{\gamma_1, \dots, \gamma_N\}$. Fix $w \in \mathbb{F}_N^+$ such that $\gamma_w(x) \in \partial \mathbb{A}_w$. Choose a sequence $(y_n)_{n \in \mathbb{N}}$ with $y_n \in \mathbb{A} \setminus \mathbb{A}_w$ and $y_n \rightarrow \gamma_w(x)$. Since $\mathbb{A} = \bigcup \{\mathbb{A}_v \mid v \in \mathbb{F}_N^+, \ell(v) = \ell(w)\}$, by passing to a subsequence we can assume that there exists $v \in \mathbb{F}_N^+$ with $\ell(v) = \ell(w)$ and $v \neq w$ such that $y_n \in \mathbb{A}_v$ for all $n \in \mathbb{N}$. Since \mathbb{A}_v is closed it follows that $\gamma_w(x) \in \mathbb{A}_v$. Let $k = \min\{l \mid v_l \neq w_l\}$. Then $\gamma_{w_k \dots w_{\ell(w)}}(x) \in \mathbb{A}_{v_k} \cap \mathbb{A}_{w_k} \subseteq C_\Gamma$. Consequently, $x \in P_\Gamma$. \square

Example 4.1.7. Consider the iterated function system $([0, 1], \Gamma)$ of Example 1.2.17. Then $P_\Gamma = [0, 1]$, so P_Γ has empty boundary. On the other hand, for each $w \in \mathbb{F}_N^+$, the set \mathbb{A}_w is a closed interval whose boundary—relative to $[0, 1]$ —consists of at most two of its endpoints. Consequently, the singular boundary ΔP_Γ is $\{0, 1\}$. \blacktriangle

Remark 4.1.8. When $\Gamma = \{\gamma\}$, injectivity of γ and Γ -invariance of \mathbb{A} imply that γ is a homeomorphism. In this case $\Delta P_\Gamma = \emptyset$.

Definition 4.1.9. An injective iterated function system (\mathbb{A}, Γ) is *post-critically stable* if there is a finite subset $F \subseteq \mathbb{F}_N^+$ such that $\Delta P_\Gamma = \bigcup_{w \in F} \gamma_w^{-1}(\partial \mathbb{A}_w)$.

In contrast the post-critically finite condition of Definition 1.2.12, a post-critically stable system does not necessarily have a finite post-critical set nor a finite singular boundary. In the following examples recall that $\mathbb{A}_i := \gamma_i(\mathbb{A})$.

Example 4.1.10. Let $\mathbb{A} = [0, 1] \times [0, 1]$ and let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, where

$$\begin{aligned} \gamma_1(x, y) &= \left(\frac{x}{2}, \frac{y}{2}\right), & \gamma_2(x, y) &= \left(\frac{1+x}{2}, \frac{y}{2}\right), \\ \gamma_3(x, y) &= \left(\frac{x}{2}, \frac{1+y}{2}\right), \text{ and } & \gamma_4(x, y) &= \left(\frac{1+x}{2}, \frac{1+y}{2}\right). \end{aligned}$$

Then (\mathbb{A}, Γ) is a post-critically stable iterated function system. The post critical set P_Γ is equal to the singular boundary ΔP_Γ , which is the perimeter of the square $[0, 1] \times [0, 1]$, as pictured in Figure 4.1. Note that (\mathbb{A}, Γ) is not post-critically finite since P_Γ is not a finite set. \triangleleft

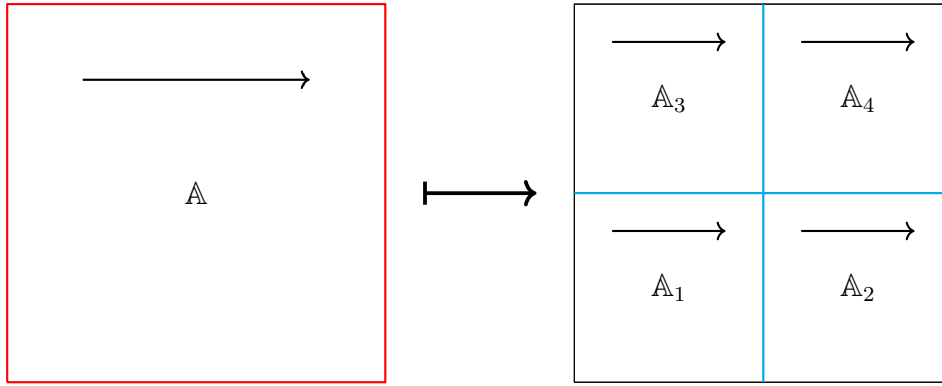


Figure 4.1: The iterated function system of Example 4.1.10. The internal arrows indicate the orientation of the squares. The singular boundary is highlighted in red, while the set $\partial \mathbb{A}_1 \cup \partial \mathbb{A}_2 \cup \partial \mathbb{A}_3 \cup \partial \mathbb{A}_4$ is blue.

Example 4.1.11. We make a modification to Example 4.1.10. Let $\mathbb{A} = [0, 1] \times [0, 1]$ and let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, where

$$\begin{aligned} \gamma_1(x, y) &= \left(\frac{2x}{3}, \frac{2y}{3}\right), & \gamma_2(x, y) &= \left(\frac{1+2x}{3}, \frac{2y}{3}\right), \\ \gamma_3(x, y) &= \left(\frac{2x}{3}, \frac{1+2y}{3}\right), \text{ and } & \gamma_4(x, y) &= \left(\frac{1+2x}{3}, \frac{1+2y}{3}\right). \end{aligned}$$

Once again (\mathbb{A}, Γ) is a post-critically stable iterated function system with singular boundary ΔP_Γ equal to the perimeter of the square $[0, 1] \times [0, 1]$ (see Figure 4.2). Unlike Example 4.1.10, the post-critical set P_Γ is the entire square $[0, 1] \times [0, 1]$, and is therefore not equal to the singular boundary. \triangleleft

Example 4.1.12. Let (\mathbb{A}, Γ) be the iterated function system from Example 1.1.11 with attractor \mathbb{A} being the Sierpinski carpet. Then $\Delta P_\Gamma = P_\Gamma$, which is pictured in Figure 1.8. \triangleleft

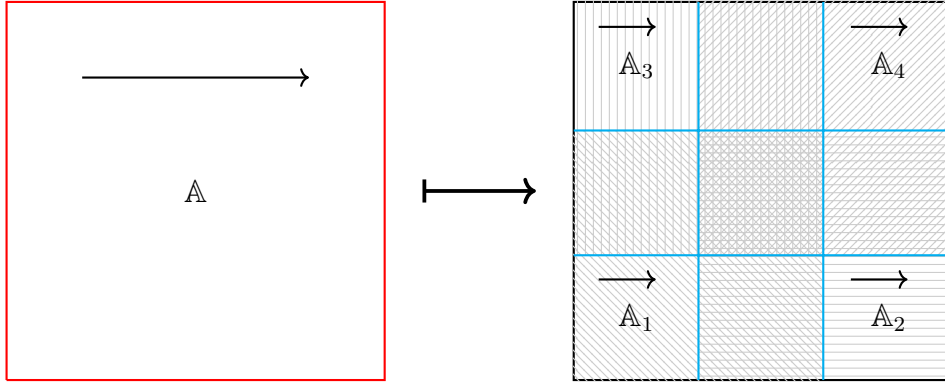


Figure 4.2: The iterated function system of Example 4.1.11. The internal arrows indicate the orientation of the squares. The singular boundary is in red, and the set $\partial A_1 \cup \partial A_2 \cup \partial A_3 \cup \partial A_4$ is blue. The 4 squares are each given a unique pattern; the cross-hatching indicates the overlap structure.

If (A, Γ) is post-critically stable, then ΔP_Γ is closed since it is a finite union of closed sets. It follows that

$$J_\Gamma := C_0(A \setminus \Delta P_\Gamma) \quad (4.8)$$

is an ideal of $A = C(A)$. For each $w \in \mathbb{F}_N^+$ define $J_w := C_0(A \setminus \gamma_w^{-1}(\partial A_w))$. If (A, Γ) is post-critically stable, then

$$J_\Gamma = \bigcap_{w \in F} J_w$$

for some finite subset $F \subseteq \mathbb{F}_N^+$. Lemma 4.1.1 yields the following.

Proposition 4.1.13. *Let (A, Γ) be a post-critically stable iterated function system. Then J_Γ is the largest ideal in $C(A)$ such that $\tau_w(a) \in C(A)$ for all $w \in \mathbb{F}_N^+$ and $a \in J_\Gamma$.*

In order to better understand J_Γ , we explore the relationship between the ideals J_w and J_v for words $w, v \in \mathbb{F}_N^+$.

Lemma 4.1.14. *For each $w, v \in \mathbb{F}_N^+$ we have $\gamma_w^{-1}(\partial A_w) \subseteq \gamma_{vw}^{-1}(\partial A_{vw})$. In particular, $J_{vw} \triangleleft J_w$.*

Proof. Suppose that $x \in A \setminus \gamma_{vw}^{-1}(\partial A_{vw})$. Then $\gamma_{vw}(x) \notin \partial A_{vw}$, so $\gamma_{vw}(x) \in \text{int}(A_{vw})$. Then $\gamma_v^{-1}(\text{int}(A_{vw}))$ is an open set containing $(\gamma_v^{-1} \circ \gamma_{vw})(x) = \gamma_w(x)$ and contained in $\gamma_v^{-1}(A_{vw}) = A_w$. Thus, $\gamma_w(x) \in \text{int}(A_w)$. So $x \notin \gamma_w^{-1}(\partial A_w)$. \square

Lemma 4.1.15. *For each $v, w \in \mathbb{F}_N^+$ we have $\gamma_{vw}^{-1}(\partial A_v) \subseteq \gamma_{vw}^{-1}(\partial A_{vw})$. Moreover, $\tau_w(J_{vw}) \subseteq J_v$.*

Proof. First observe that $\gamma_{vw}^{-1}(\partial A_v) = \gamma_{vw}^{-1}(\partial A_v \cap A_{vw})$. Thus, it suffices to show that $\partial A_v \cap A_{vw} \subseteq \partial A_{vw}$. Since $A_{vw} \subseteq A_v$ we have $\partial A_v \cap A_{vw} = \overline{A_v \cap A_{vw}} \setminus A_v \cap A_{vw} = \overline{A} \setminus \overline{A_v \cap A_{vw}}$. Using Γ -invariance of A we also have $A_v = \bigcup \{A_{v\mu} \mid \mu \in \mathbb{F}_N^+, \ell(\mu) = \ell(w)\}$. Consequently,

$$\begin{aligned} \overline{A} \setminus \overline{A_v} \cap A_{vw} &= A_{vw} \cap \overline{\{A \setminus A_{v\mu} \mid \mu \in \mathbb{F}_N^+, \ell(\mu) = \ell(w)\}} \\ &\subseteq A_{vw} \cap \overline{\{A \setminus A_{v\mu} \mid \mu \in \mathbb{F}_N^+, \ell(\mu) = \ell(w)\}} \\ &\subseteq A_{vw} \cap \overline{A \setminus A_{vw}} \\ &= \partial A_{vw}. \end{aligned}$$

Hence, $\partial\mathbb{A}_v \cap \mathbb{A}_{vw} \subseteq \partial\mathbb{A}_{vw}$.

For the second statement observe that the first statement implies that $\mathbb{A} \setminus \gamma_{vw}^{-1}(\partial\mathbb{A}_{vw}) \subseteq \mathbb{A} \setminus \gamma_{vw}^{-1}(\partial\mathbb{A}_v)$. In particular, $J_{vw} \subseteq C_0(\mathbb{A} \setminus \gamma_w^{-1}(\gamma_v^{-1}(\partial\mathbb{A}_v)))$. Now suppose that $a \in J_{vw} \triangleleft J_w$ and $x \in \gamma_v^{-1}(\partial\mathbb{A}_v)$. Then

$$\begin{aligned} \tau_w(a)(x) &= \begin{cases} a(\gamma_w^{-1}(x)) & \text{if } x \in \mathbb{A}_w, \\ 0 & \text{otherwise,} \end{cases} \\ &= 0. \end{aligned}$$

Since $\tau_w(a)$ is continuous and vanishes on $\gamma_v^{-1}(\partial\mathbb{A}_v)$, it follows that $\tau_w(a) \in J_v$. \square

Recall that we write Γ for the Hutchinson operator (see Equation (1.3)). In particular,

$$\Gamma(\Delta P_\Gamma) = \bigcup_{\gamma \in \Gamma} \gamma(\Delta P_\Gamma).$$

Lemma 4.1.16. *Let $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ be a injective iterated function system. If $x \in \mathbb{A}_w \cap \Delta P_\Gamma$, then $\gamma_w^{-1}(x) \in \Delta P_\Gamma$.*

Proof. Suppose that $x \in \gamma_v^{-1}(\partial\mathbb{A}_v) \cap \mathbb{A}_w$. Lemma 4.1.15 gives

$$\gamma_w^{-1}(x) \in \gamma_{vw}^{-1}(\partial\mathbb{A}_v) \subseteq \gamma_{vw}^{-1}(\partial\mathbb{A}_{vw}) \subseteq \Delta P_\Gamma.$$

For the second statement let $x \in \Delta P_\Gamma$ and take $1 \leq i \leq N$ such that $x \in \mathbb{A}_i$. Then $x = \gamma_i \circ \gamma_i^{-1}(x) \in \Gamma(\Delta P_\Gamma)$ by the first statement. \square

There are post-critically stable systems whose singular boundary is the entire attractor.

Example 4.1.17. Let C denote the middle thirds cantor set. Define an injective iterated function system $(\mathbb{A} := C \times C, \Gamma = \{\gamma_1, \gamma_2\})$, where $\gamma_1 = \text{id}_{C \times C}$ and γ_2 is a homeomorphism from $C \times C$ onto $C \times \{0\}$ composed with the inclusion of $C \times \{0\}$ into $C \times C$. Since $C \times \{0\}$ has empty interior, $\mathbb{A}_2 = \partial\mathbb{A}_2$. In particular, $\Delta P_\Gamma = \gamma_2^{-1}(\partial\mathbb{A}_2) = \mathbb{A}$. For this system Lemma 4.1.1 implies $\tau_2(a)$ is discontinuous for all $a \in C(\mathbb{A})$. \blacktriangle

In all the examples we are currently interested in, $\Delta P_\Gamma \neq \mathbb{A}$. In fact, ΔP_Γ often has empty interior. We identify such systems.

Definition 4.1.18. A post-critically stable iterated function system (\mathbb{A}, Γ) is said to have *regular overlap* if ΔP_Γ has empty interior.

Example 4.1.19. The systems from Examples 4.1.10 and 4.1.11 have regular overlap. \blacktriangle

4.2 | The lacunary algebra

In this section we introduce a new C^* -algebra associated to an iterated function system, that we call the *lacunary algebra*. We assume throughout this section that (\mathbb{A}, Γ) is an injective, post-critically stable iterated function system. The lacunary algebra is constructed as the C^* -algebra of a topological graph, which were discussed in Chapter 3. To this end, let

- $E^0 := \mathbb{A} \setminus \Delta P_\Gamma$;
- $E^1 := (\mathbb{A} \setminus \Delta P_\Gamma) \times \Gamma$;
- $s(x, \gamma) := x$ for all $(x, \gamma) \in E^1$; and
- $r(x, \gamma) := \gamma(x)$ for all $(x, \gamma) \in E^1$.

Clearly s is a local homeomorphism. By Lemma 4.1.16, if $\gamma \in \Gamma$, then $\gamma^{-1}(\Delta P_\Gamma) \subseteq \Delta P_\Gamma$, and so we have $r(E^1) \subseteq \mathbb{A} \setminus \Delta P_\Gamma$. As such, r defines a continuous map $r: E^1 \rightarrow E^0$ and so $E = (E^0, E^1, r, s)$ is a topological graph.

Definition 4.2.1. Recall that $J_\Gamma = C_0(\mathbb{A} \setminus \Delta P_\Gamma) = C_0(E^0)$. Denote the J_Γ - J_Γ -correspondence associated to the topological graph E by Y_Γ . We call Y_Γ the *lacunary correspondence* associated to (\mathbb{A}, Γ) . We call the Cuntz-Pimsner algebra \mathcal{O}_Y the *lacunary algebra* associated to (\mathbb{A}, Γ) .

Since J_Γ is usually non-unital, \mathcal{O}_{Y_Γ} is usually non-unital. Following Section 3.1.2 we describe the correspondence Y_Γ explicitly. The right J_Γ -action on Y_Γ is defined on $C_c(E^1)$ by

$$(\xi \cdot a)(x, \gamma) = \xi(x, \gamma)a(s(x, \gamma)) = \xi(x, \gamma)a(x)$$

for all $\xi \in C_c(E^1)$ and $a \in J_\Gamma$, and the J_Γ -valued inner product satisfies

$$(\xi \mid \eta)_{J_\Gamma}(x) = \sum_{\gamma \in \Gamma} \overline{\xi(x, \gamma)} \eta(x, \gamma)$$

for all $\xi, \eta \in C_c(E^1)$. The left action of J_Γ on Y_Γ is given by

$$(a \cdot \xi)(x, \gamma) = a(r(x, \gamma))\xi(x, \gamma) = a(\gamma(x))\xi(x, \gamma)$$

for all $a \in J_\Gamma$ and $\xi \in C_c(E^1)$. We write ϕ for the $*$ -homomorphism $\phi: J_\Gamma \rightarrow \text{End}_J^0(Y_\Gamma)$ implementing the left action.

For each $\gamma \in \Gamma$, observe that the module norm on $C_c((\mathbb{A} \setminus \Delta P_\Gamma) \times \{\gamma\}) \subseteq Y_\Gamma$ is just the supremum norm. Let $Y_\gamma = C_0((\mathbb{A} \setminus \Delta P_\Gamma) \times \{\gamma\})$ and note that Y_γ is isomorphic—as a right J_Γ -module—to J_Γ . Moreover, Y_γ can be equipped with a left J_Γ -action induced by the map γ . Then Y_Γ is isomorphic to $\bigoplus_{\gamma \in \Gamma} Y_\gamma$ as correspondences over J_Γ . It follows that the norm on Y_Γ is equivalent to the supremum norm on $C_0(E^1)$. As a right J_Γ -module Y_Γ can also be identified with $X_\Gamma \cdot J_\Gamma = \overline{\text{span}}\{\xi \cdot a \mid \xi \in X_\Gamma, a \in J_\Gamma\}$.

Proposition 4.2.2. *Suppose that (\mathbb{A}, Γ) is an injective iterated function system admitting a code map, and $\Delta P_\Gamma = \emptyset$. Then \mathcal{O}_Y is isomorphic to the Cuntz algebra $\mathcal{O}_{|\Gamma|}$.*

Proof. If $\Delta P_\Gamma = \emptyset$, then $J_\Gamma = C(\mathbb{A})$, in which case $X_\Gamma \cong Y_\Gamma$. It then follows from Proposition 2.1.4 that $\mathcal{O}_Y \cong \mathcal{O}_{|\Gamma|}$. \square

Remark 4.2.3. The construction of Y_Γ could be repeated using $E^1 = \text{Gr}(\Gamma) \setminus p_2^{-1}(\Delta P_\Gamma)$, to arrive at a submodule of the Kajiwara-Watatani module E_Γ (see Chapter 2). One technical issue with this approach is that branched points would once again be introduced. Since we are currently interested in studying the overlap properties of (\mathbb{A}, Γ) , the introduction of branched points is unnecessary.

Our immediate goal is to compute the K -theory of \mathcal{O}_Y . To this end, we need to identify the Katsura ideal $I_Y := \phi^{-1}(\text{End}_J^0(Y_\Gamma))$ of (ϕ, Y_Γ) . Recall that finite receivers of E are given by

$$E_{fr}^0 := \{x \in E^0 \mid \text{there exists a precompact open neighbourhood } W \text{ of } x \text{ such that } r^{-1}(\overline{W}) \text{ is compact}\},$$

and the sources of E are given by

$$E_{src}^0 := E^0 \setminus \overline{r(E^1)}.$$

Since Y_Γ is defined by a topological graph construction, it follows from Proposition 3.1.10 that the Katsura ideal is $I_Y = C_0(E_{fr}^0 \setminus \overline{E_{src}^0})$. Thus, our aim is to compute both E_{src}^0 and E_{fr}^0 .

Lemma 4.2.4. *If (\mathbb{A}, Γ) has regular overlap then $E_{src}^0 = \emptyset$.*

Proof. Observe that $r(E^1) = \bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A} \setminus \Delta P_\Gamma)$. Since ΔP_Γ has no interior $\mathbb{A} \setminus \Delta P_\Gamma$ is dense in \mathbb{A} . It follows that $\gamma(\mathbb{A} \setminus \Delta P_\Gamma)$ is dense in $\gamma(\mathbb{A})$. Now suppose that W is an open subset of \mathbb{A} . Then Γ -invariance of \mathbb{A} implies that there exists $\gamma \in \Gamma$ such that $W \cap \gamma(\mathbb{A})$ is non-empty. It follows that $W \cap \gamma(\mathbb{A} \setminus \Delta P_\Gamma)$ is non-empty, so $r(E^1)$ is dense in E^0 . Hence, $\overline{r(E^1)} = E^0$. \square

Lemma 4.2.5. *Suppose that (\mathbb{A}, Γ) is a post-critically stable iterated function system. Then $E_{fr}^0 = \mathbb{A} \setminus \Gamma(\Delta P_\Gamma)$.*

Proof. Recall from Lemma 4.1.16 that $\Gamma(\Delta P_\Gamma) \subseteq \Delta P_\Gamma$, so $\mathbb{A} \setminus \Gamma(\Delta P_\Gamma) = E^0 \setminus \Gamma(\Delta P_\Gamma)$. Let $x \in E^0 \setminus \Gamma(\Delta P_\Gamma)$ and take a precompact open neighbourhood W of x in E^0 . Then

$$r^{-1}(\overline{W}) = \bigcup_{\gamma \in \Gamma} \gamma^{-1}(\overline{W} \cap \gamma(\mathbb{A})) \times \{\gamma\}.$$

Since $\overline{W} \cap \gamma(\mathbb{A})$ is compact in E^0 , it is also compact in \mathbb{A} . Then $\gamma^{-1}(\overline{W} \cap \gamma(\mathbb{A}))$ is contained in E^0 , and compact in \mathbb{A} , because \mathbb{A} is compact. Since E^0 is open in \mathbb{A} , $\gamma^{-1}(\overline{W} \cap \gamma(\mathbb{A}))$ is compact in E^0 . Hence, $r^{-1}(\overline{W})$ is compact in E^1 .

Now suppose that $x \in \Gamma(\Delta P_\Gamma) \setminus \Delta P_\Gamma$. Then there exists $\gamma \in \Gamma$ such that $x \in \gamma(\mathbb{A})$ and $\gamma^{-1}(x) \in \Delta P_\Gamma$. Fix a precompact open neighbourhood $W \subseteq E^0$ of x . We claim that $r^{-1}(\overline{W})$ is not compact in E^1 . Fix a sequence $(x_n)_{n=1}^\infty$ in $\gamma(\mathbb{A}) \cap W$ such that $x_n \rightarrow x$. Then $\gamma^{-1}(x_n) \rightarrow \gamma^{-1}(x)$ in \mathbb{A} . Since $\gamma^{-1}(x) \in \Delta P_\Gamma$, it follows that $(\gamma^{-1}(x_n))_{n=1}^\infty$ has no subsequence that converges in E^0 . Since $((\gamma^{-1}(x_n), \gamma))_{n=1}^\infty$ is a sequence in $r^{-1}(\overline{W})$ with no convergent subsequence, $r^{-1}(\overline{W})$ is not compact. \square

We can now identify the Katsura ideal I_Y of Y_Γ .

Proposition 4.2.6. *If (\mathbb{A}, Γ) is an injective iterated function system with regular overlap, then Katsura ideal I_Y of (ϕ, Y_Γ) is $I_Y = C_0(\mathbb{A} \setminus \Gamma(\Delta P_\Gamma))$.*

Proof. This follows from Proposition 3.1.10, Lemma 4.2.4, and Lemma 4.2.5. \square

To finish this section note that by virtue of being a topological graph algebra, \mathcal{O}_Y can be realised as a groupoid C^* -algebra, and so the discussion in Section 3.1.4 applies. We

keep the description of the groupoid model for \mathcal{O}_Y brief since we do not use it explicitly. To this end, note that the boundary path space (Definition 3.1.19) of E is given by,

$$\partial E = E^\infty \sqcup \bigsqcup_{k=0}^{\infty} \{x \in E^k \mid s(x) \in \Gamma(\partial P_\Gamma)\}.$$

Define $\sigma: E^{\leq \infty} \rightarrow E^{\leq \infty} \setminus E^0$ by $\sigma(x) = x_{[2,k]}$ for all $x \in E^k$ and $1 \leq k \leq \infty$. Lemma 3.1.20 implies that σ restricts to a partial local homeomorphism $\sigma: \partial E \setminus \Gamma(\Delta P_\Gamma) \rightarrow \partial E$. Let $\text{dom}(\sigma^n) := \partial E \setminus \{x \in \bigsqcup_{k=0}^{n-1} E^k \mid s(x) \in \Gamma(\Delta P_\Gamma)\}$. Since $(\partial E, \sigma)$ is a singly generated dynamical system, we can form the Deaconu-Renault groupoid

$$G(\partial E, \sigma) := \{(x, m - n, y) \mid m, n \in \mathbb{N}_0, x \in \text{dom}(\sigma^m), y \in \text{dom}(\sigma^n), \sigma^m(x) = \sigma^n(y)\}.$$

The following is then an immediate consequence of Theorem 3.1.22.

Proposition 4.2.7. *Let (\mathbb{A}, Γ) be an injective iterated function system with regular overlap. Then $C^*(G(\partial E, \sigma)) \cong \mathcal{O}_Y$.*

4.3 | The K -theory of \mathcal{O}_Y

We now consider the K -theory of \mathcal{O}_Y , and investigate what data it contains about the dynamics and topology of (\mathbb{A}, Γ) . Once we have developed some tools for computing the K -theory of \mathcal{O}_Y , we apply the machinery to a number of examples.

As a first step towards computing the K -theory of \mathcal{O}_Y we have the following application of Theorem A.3.17.

Corollary 4.3.1. *Let (\mathbb{A}, Γ) be an injective iterated function system. Let $\iota_{I,J}: I_Y \rightarrow J_\Gamma$ be the inclusion map, and let $\pi: J_\Gamma \rightarrow \mathcal{O}_Y$ be the universal inclusion. Then there is a six-term exact sequence, as follows:*

$$\begin{array}{ccccc} K_0(I_Y) & \xrightarrow{\otimes(\iota_{I,J*} - [Y_\Gamma])} & K_0(J_\Gamma) & \xrightarrow{\pi_*} & K_0(\mathcal{O}_Y) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathcal{O}_Y) & \xleftarrow{\pi_*} & K_1(J_\Gamma) & \xleftarrow{\otimes(\iota_{I,J*} - [Y_\Gamma])} & K_1(I_Y). \end{array}$$

To use the six-term sequence of Corollary 4.3.1, we require a better understanding of the class $[Y_\Gamma] \in KK(I_Y, J_\Gamma)$. Recall that Y_Γ is a direct sum $\bigoplus_{\gamma \in \Gamma} Y_\gamma$ of J_Γ - J_Γ -correspondences. Thus, $[Y_\Gamma] = \sum_{\gamma \in \Gamma} [Y_\gamma]$ in $KK(I_Y, J_\Gamma)$, reducing our problem to understanding $[Y_\gamma]$ for each $\gamma \in \Gamma$. We aim to describe $[Y_\gamma]$ in terms of the map γ .

The left action of J_Γ on Y_γ of course depends on γ . Indeed, if $a \in J_\Gamma$ and $\xi \in Y_\gamma$, then $(a \cdot \xi)(x, \gamma) = a(\gamma(x))\xi(x, \gamma)$. Consequently, if $a \in I_Y$ is such that $\text{supp}(a) \cap \gamma(\mathbb{A}) = \emptyset$, then a is in the kernel of the left action on Y_γ .

Lemma 4.3.2. *For each $\gamma \in \Gamma$, the set $\gamma(\mathbb{A}) \setminus \Gamma(\Delta P_\Gamma)$ is clopen in $\mathbb{A} \setminus \Gamma(\Delta P_\Gamma)$. In particular, $I_\gamma := C_0(\gamma(\mathbb{A}) \setminus \Gamma(\Delta P_\Gamma))$ satisfies $I_Y = I_\gamma \oplus I_\gamma^\perp$.*

Proof. Since $\gamma(\mathbb{A})$ is closed in \mathbb{A} , it follows that $\gamma(\mathbb{A}) \setminus \Gamma(\Delta P_\Gamma)$ is closed in $\mathbb{A} \setminus \Gamma(\Delta P_\Gamma)$. As $\partial(\gamma(\mathbb{A})) = \gamma \circ \gamma^{-1}(\partial(\gamma(\mathbb{A}))) \subseteq \Gamma(\Delta P_\Gamma)$, it follows that $\gamma(\mathbb{A}) \setminus \Gamma(\Delta P_\Gamma) = \text{int}(\gamma(\mathbb{A})) \setminus \Gamma(\Delta P_\Gamma)$. Thus, $\gamma(\mathbb{A}) \setminus \Gamma(\Delta P_\Gamma)$ is open in $\mathbb{A} \setminus \Gamma(\Delta P_\Gamma)$. \square

By Lemma 4.3.2, the kernel of the left action of J_Γ on Y_γ is I_γ^\perp . Since $I_Y = I_\gamma \oplus I_\gamma^\perp$, restriction of functions $r_\gamma: I_\Gamma \rightarrow I_\gamma$ is a $*$ -homomorphism. We can therefore restrict the left action of I_Y on Y_γ to I_γ .

Let $\bar{\gamma}$ denote the restriction of γ to $\mathbb{A} \setminus \Delta P_\Gamma$. Then $\bar{\gamma}(\mathbb{A} \setminus \Delta P_\Gamma) = \gamma(\mathbb{A}) \setminus \gamma(\Delta P_\Gamma)$. Since $\partial(\gamma(\mathbb{A})) = \gamma \circ \gamma^{-1}(\partial(\gamma(\mathbb{A}))) \in \gamma(\Delta P_\Gamma)$, it follows that $\gamma(\mathbb{A}) \setminus \gamma(\Delta P_\Gamma)$ is open in \mathbb{A} . We claim that $\bar{\gamma}$ is proper. Suppose that K is a compact subset of $\gamma(\mathbb{A}) \setminus \gamma(\Delta P_\Gamma)$. Since $\gamma(\mathbb{A}) \setminus \gamma(\Delta P_\Gamma)$ is open, K is compact in \mathbb{A} . Moreover, $\bar{\gamma}^{-1}(K) = \gamma^{-1}(K)$ is compact in \mathbb{A} , and therefore compact in $\mathbb{A} \setminus \Delta P_\Gamma$. Thus $\bar{\gamma}$ induces a dual $*$ -homomorphism $\bar{\gamma}^*: C_0(\gamma(\mathbb{A}) \setminus \gamma(\Delta P_\Gamma)) \rightarrow J_\Gamma$. Let $\alpha_\gamma: I_\gamma \rightarrow J_\Gamma$ denote the restriction of $\bar{\gamma}^*$ to $I_\gamma \subseteq C_0(\gamma(\mathbb{A}) \setminus \gamma(\Delta P_\Gamma))$.

Proposition 4.3.3. *The I_γ - J_Γ -correspondence associated to the $*$ -homomorphism α_γ is isomorphic to Y_γ . In particular, the Kasparov product $\cdot \otimes [Y_\gamma]: K_*(I_Y) \rightarrow K_*(J_\Gamma)$ is equal to the map induced on K -theory by $\alpha_\gamma \circ r_\gamma$. In particular, the Kasparov product $\cdot \otimes [Y_\Gamma]: K_*(I_Y) \rightarrow K_*(J_\Gamma)$ is given by $\sum_{\gamma \in \Gamma} (\alpha_\gamma \circ r_\gamma)_*$.*

Proof. The proof follows directly from the definitions of the maps involved. \square

Before we move on to examples, we recall the following facts about the K -theory of commutative C^* -algebras defined on planar sets.

Proposition 4.3.4 ([HR00, Proposition 7.5.2]). *If X is a non-empty compact subset of \mathbb{C} , then the character homomorphisms from $K_n(C(X))$ to the Čech cohomology groups $\check{H}^n(X)$ are isomorphisms for $n = 0, 1$.*

Proposition 4.3.5 ([HR00, Proposition 7.5.3]). *Let X be a non-empty compact subset of \mathbb{C} . Let $\{\lambda_1, \lambda_2, \dots\}$ be a sequence of points in $\mathbb{C} \setminus X$ such that each bounded component of $\mathbb{C} \setminus X$ contains precisely one λ_i , and no λ_i is contained in the unbounded component. Then $\check{H}^1(X)$ is the free Abelian group generated by the homotopy classes of the functions $z \mapsto z - \lambda_j$ on X .*

4.3.1 Interval maps

As in Section 2.4.1 we again consider iterated function systems on $[0, 1]$. Let $([0, 1], \Gamma)$ be an injective iterated function system. Since each $\gamma \in \Gamma$ is injective, $\gamma([0, 1])$ is a closed interval in $[0, 1]$. With boundaries computed in the relative topology on $[0, 1]$, it follows that for each $\gamma \in \Gamma$, $\partial\gamma([0, 1])$ consists of either 0, 1, or 2 endpoints of the interval $\gamma([0, 1])$. It follows that the singular boundary ΔP_Γ consists of either 0, 1, or 2 of the endpoints of $[0, 1]$. We consider each of these cases individually. Before we begin, recall from Lemma 4.1.16 that $\Gamma(\Delta P_\Gamma) \subseteq \Delta P_\Gamma$.

$\Delta P_\Gamma = \emptyset$. If $\Delta P_\Gamma = \emptyset$ then $J_\Gamma = C([0, 1]) = A$. Accordingly, $Y_\Gamma = X_\Gamma$. Since $K_0(C([0, 1])) = \mathbb{Z}[1_A]$ and $K_1(C([0, 1])) = 0$, Corollary 4.3.1 implies that

$$K_0(\mathcal{O}_Y) \cong \text{coker}(\otimes(\text{id}_A - [Y_\Gamma])) \quad \text{and} \quad K_1(\mathcal{O}_Y) \cong \ker(\otimes(\text{id}_A - [Y_\Gamma])).$$

Proposition 4.3.3 now shows that $[1_A] \otimes [Y_\Gamma] = |\Gamma|[1_A]$. Consequently,

$$K_0(\mathcal{O}_Y) \cong \mathbb{Z}/(1 - |\Gamma|)\mathbb{Z} \quad \text{and} \quad K_1(\mathcal{O}_Y) = 0.$$

If (\mathbb{A}, Γ) admits a code map, then Proposition 4.2.2 implies that \mathcal{O}_Y is isomorphic to the Cuntz algebra $\mathcal{O}_{|\Gamma|}$.

Example 4.3.6. Let $\Gamma = \{\gamma_1, \gamma_2\}$ with $\gamma_1(x) = x$ and $\gamma_2(x) = 1 - x$. Then $\Delta P_\Gamma = \emptyset$, so $K_0(\mathcal{O}_Y) = 0$ and $K_1(\mathcal{O}_Y) = 0$. \triangle

$|\Delta P_\Gamma| = 1$. If $|\Delta P_\Gamma| = 1$, then J_Γ is equal to either $C_0([0, 1))$ or $C_0((0, 1])$. Hence, $K_0(J_\Gamma) = K_1(J_\Gamma) = 0$. It now follows from Corollary 4.3.1 that $K_0(I_Y) \cong K_1(\mathcal{O}_Y)$ and $K_1(I_Y) \cong K_0(\mathcal{O}_Y)$.

Let $M \leq |\Gamma|$ denote the number of distinct points in $\Gamma(\Delta P_\Gamma) \setminus \Delta P_\Gamma$. Since $\Gamma(\Delta P_\Gamma) \subseteq \Delta P_\Gamma$, the connected components of $[0, 1] \setminus \Gamma(\Delta P_\Gamma)$ consist of precisely M open intervals, with at most one half-open interval. In particular, $K_0(I_Y) = 0$ and $K_1(I_Y) \cong \mathbb{Z}^M$. Hence, $K_0(\mathcal{O}_Y) \cong \mathbb{Z}^M$ and $K_1(\mathcal{O}_Y) = 0$. So $K_0(\mathcal{O}_Y)$ remembers some of the overlap structure of (\mathbb{A}, Γ) by means of the number M .

Example 4.3.7. Let $\Gamma = \{\gamma_1, \gamma_2\}$ be given by $\gamma_1(x) = x/2$ and $\gamma_2(x) = x$. Then $\Delta P_\Gamma = \{1\}$ and $\Gamma(\Delta P_\Gamma) = \{1/2, 1\}$. It follows that $K_0(\mathcal{O}_Y) \cong \mathbb{Z}$ and $K_1(\mathcal{O}_Y) = 0$. \triangle

As the following result shows, examples of iterated function systems on $[0, 1]$ for which $|\Delta P_\Gamma| = 1$ are always of a restricted form.

Proposition 4.3.8. *Let $([0, 1], \Gamma = \{\gamma_1, \dots, \gamma_N\})$ be an injective iterated function system with $|\Delta P_\Gamma| = 1$. Let $1 \leq i, j \leq N$ be such that $0 \in \mathbb{A}_i$ and $1 \in \mathbb{A}_j$. Then at least one of i or j is surjective, and hence a homeomorphism.*

Proof. Without loss of generality suppose that $\Delta P_\Gamma = \{1\}$. For contradiction, suppose that each γ_i and γ_j are not surjective. Since γ_i is injective, we must either have $\gamma_i(1) = 0$ or $\gamma_i(0) = 0$, as γ_i is a homeomorphism onto its image.

First suppose that $\gamma_i(1) = 0$. Then $\gamma_i(0) \in \partial \mathbb{A}_i$, for if $\gamma_i(0) = 1$, continuity of γ_i would imply that γ_i is surjective. However, if $\gamma_i(0) \in \partial \mathbb{A}_i$, then $0 = \gamma_i^{-1} \circ \gamma_i(0) \in \Delta P_\Gamma$, which defies our assumption that $\Delta P_\Gamma = \{1\}$. Hence, it cannot be the case that $\gamma_i(1) = 0$, and so we have $\gamma_i(0) = 0$ and $\gamma_i(1) \in \partial \mathbb{A}_i$.

A similar argument to above implies that $\gamma_j(0) = 1$. Since $1 \in \gamma_i^{-1}(\partial \mathbb{A}_i)$, it follows from Lemma 4.1.15 that

$$0 = \gamma_j^{-1}(1) \in \gamma_j^{-1}(\gamma_i^{-1}(\partial \mathbb{A}_i)) = \gamma_{ij}^{-1}(\partial \mathbb{A}_i) \subseteq \gamma_{ij}^{-1}(\partial \mathbb{A}_{ij}) \subseteq \Delta P_\Gamma.$$

This contradicts the fact that $\Delta P_\Gamma = \{1\}$. Hence, either γ_i or γ_j must be surjective. \square

$\Delta P_\Gamma = \{0, 1\}$. If $\Delta P_\Gamma = \{0, 1\}$, then $J_\Gamma \cong C_0((0, 1))$. Hence, $K_0(J_\Gamma) = 0$ and $K_1(J_\Gamma) \cong \mathbb{Z}$. Let $N \leq 2|\Gamma|$ denote the number of distinct points in $\Gamma(\Delta P_\Gamma) \setminus \Delta P_\Gamma$. The connected components of $[0, 1] \setminus \Gamma(\Delta P_\Gamma)$ consist of precisely $N + 1$ open intervals. Accordingly, $K_0(I_Y) = 0$ and $K_1(I_Y) \cong \mathbb{Z}^{N+1}$. It now follows from Corollary 4.3.1 that

$$K_0(\mathcal{O}_Y) \cong \ker(\otimes(\iota_{I, J*} - [Y_\Gamma])) \quad \text{and} \quad K_1(\mathcal{O}_Y) \cong \text{coker}(\otimes(\iota_{I, J*} - [Y_\Gamma])).$$

Since $K_1(J_\Gamma) \cong \mathbb{Z}$, we find that $K_0(\mathcal{O}_Y)$ is isomorphic to either \mathbb{Z}^N or \mathbb{Z}^{N+1} , depending on the surjectivity of $\cdot \otimes(\iota_{I, J*} - [Y_\Gamma])$.

To determine the map $\cdot \otimes [Y_\Gamma]: K_1(I_Y) \rightarrow K_1(\Gamma)$, let $v \in C(\mathbb{T})$ denote a unitary generator of $K_1(C_0((0, 1))) \cong K_1(C(\mathbb{T}))$. For $1 \leq n \leq N$ let U_n denote the open intervals making up the connected components of $[0, 1] \setminus \Gamma(\Delta P_\Gamma)$. Choose unitary generators $u_n \in C(\mathbb{T})$ of $K_1(C_0(U_n)) \cong K_1(C(\mathbb{T}))$ such that the map $\iota_{I, J*}$ —induced by the inclusion of I_Y into J_Y —takes $[u_n]$ to $[v]$ for each $n \in \mathbb{N}$.

We use Proposition 4.3.3 to compute $[u_n] \otimes [Y_\Gamma]$. Injectivity implies that each $\gamma \in \Gamma$ is either strictly increasing or decreasing. For each $\gamma \in \Gamma$ let

$$\delta(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ is strictly increasing,} \\ 0 & \text{if } \gamma \text{ is strictly decreasing.} \end{cases}$$

Suppose that $\gamma \in \Gamma$ satisfies $U_n \subseteq \gamma(\mathbb{A})$ and let $\alpha_\gamma: I_\gamma \rightarrow J_\gamma$ be as in Proposition 4.3.3. If $\delta(\gamma) = 1$, then the map induced by α_γ on $C(\mathbb{T})$ preserves the orientation of unitaries and takes u_n to v , while if $\delta(\gamma) = -1$, then u_n is taken to \bar{v} . Hence, for all $\gamma \in \Gamma$ and $U_n \subseteq \gamma(\mathbb{A})$, we have $(\alpha_\gamma)_*([u_n]) = \delta(\gamma)[v]$. Proposition 4.3.3 implies

$$[u_n] \otimes [Y_\Gamma] = \sum_{\gamma \in \Gamma} [u_n] \otimes [Y_\gamma] = \sum_{\{\gamma \in \Gamma \mid U_n \subseteq \gamma(\mathbb{A})\}} \delta(\gamma)[v].$$

Hence,

$$[u_n] \otimes (\iota_{I,J*} - [Y_\Gamma]) = [v] - \sum_{\{\gamma \in \Gamma \mid U_n \subseteq \gamma(\mathbb{A})\}} \delta(\gamma)[v]. \quad (4.9)$$

The cokernel of $\cdot \otimes (\iota_{I,J*} - [Y_\Gamma])$ can be found using Smith normal forms (see [Sta16, Theorem 2.3]). By [Sta16, Theorem 2.4],

$$K_1(\mathcal{O}_Y) \cong \frac{\mathbb{Z}}{\gcd(\{1 - \sum_{\{\gamma \in \Gamma \mid U_n \subseteq \gamma(\mathbb{A})\}} \delta(\gamma) \mid 1 \leq n \leq N\})\mathbb{Z}},$$

with the convention that $\gcd(\{0\}) = 0$, and

$$K_0(\mathcal{O}_Y) \cong \begin{cases} \mathbb{Z}^N & \text{if } K_1(\mathcal{O}_Y) \neq \mathbb{Z}, \\ \mathbb{Z}^{N+1} & \text{if } K_1(\mathcal{O}_Y) = \mathbb{Z}. \end{cases}$$

The number N appearing in $K_0(\mathcal{O}_Y)$ depends on the overlap structure of (\mathbb{A}, Γ) . On the other hand, $K_1(\mathcal{O}_Y)$ contains data about the cumulative orientation of the maps $\gamma \in \Gamma$.

Example 4.3.9. Let $\Gamma = \{\gamma_1, \gamma_2\}$ with $\gamma_1(x) = \frac{x}{2}$ and $\gamma_2(x) = \frac{x}{2} + \frac{1}{2}$; let $\Gamma' = \{\gamma'_1, \gamma'_2\}$ with $\gamma'_1(x) = \frac{2x}{3}$ and $\gamma'_2(x) = \frac{2x}{3} + \frac{1}{3}$; let $\Gamma'' = \{\gamma''_1, \gamma''_2\}$ with $\gamma''_1(x) = 1 - \frac{x}{2}$ and $\gamma''_2(x) = \frac{1}{2} - \frac{x}{2}$; and let $\Gamma''' = \{\gamma'''_1, \gamma'''_2\}$. Then $\Gamma(\Delta P_\Gamma) = \Gamma''(\Delta P_{\Gamma''}) = \Gamma'''(\Delta P_{\Gamma'''}) = \{0, \frac{1}{2}, 1\}$ and $\Gamma'(\Delta P_{\Gamma'}) = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. Tabulating the resulting K -groups yields

	$K_0(\mathcal{O}_Y)$	$K_1(\mathcal{O}_Y)$
Γ	\mathbb{Z}^2	\mathbb{Z}
Γ'	\mathbb{Z}^2	0
Γ''	\mathbb{Z}	$\mathbb{Z}/2\mathbb{Z}$
Γ'''	\mathbb{Z}	$\mathbb{Z}/2\mathbb{Z}$.

There are a few important things observe about the preceding calculation. Proposition 2.4.2 implies that the K -groups for Kajiwara-Watatani algebras associated to these three systems agree. Since Γ , Γ' , and Γ'' have no branched points and are contractive, it follows from Proposition 2.1.4 that the associated Kajiwara-Watatani algebras are isomorphic to the Cuntz algebra \mathcal{O}_2 .

By restricting the coefficient algebra to the ideal $C_0((0, 1))$ we are able to distinguish between these examples. Comparing the K_0 -groups for Γ and Γ' we see that the overlap structure plays a critical role. On the other hand, comparing Γ to Γ'' , the K_1 -group picks up information related to the orientation preserving or reversing properties of the maps.

On the other hand, we note that K -groups alone do not distinguish between Γ'' and Γ''' . Let $U_1 = (0, \frac{1}{2})$ and $U_2 = (\frac{1}{2}, 1)$, and let u_1 and u_2 be the associated unitaries generating $K_1(C_0(U_1))$ and $K_1(C_0(U_2))$, respectively. Then $\ker(\iota_{I,J*} - [Y_{\Gamma''}])$ is generated by $[u_1] - [u_2]$, while $\ker(\iota_{I,J*} - [Y_{\Gamma'''}])$ is generated by $[u_1]$. Consequently, it may be possible to distinguish between the two examples with the additional data of index pairings. \blacktriangle

4.3.2 Sierpinski gasket

Let \blacktriangle denote the Sierpinski gasket with vertices $(0, 0)$, $(1/2, \sqrt{3}/2)$, and $(1, 0)$. Recall from Example 1.1.10 that \blacktriangle is the attractor of the contractive iterated function system $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ on \mathbb{R}^2 , where

$$\gamma_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right), \quad \gamma_2(x, y) = \left(\frac{x+1}{4}, \frac{2y+\sqrt{3}}{4}\right), \quad \text{and} \quad \gamma_3(x, y) = \left(\frac{x+1}{2}, \frac{y}{2}\right).$$

We compute the K -theory of \mathcal{O}_{Y_Γ} for this system using Corollary 4.3.1. To do so, we compute the K -theory of I_Y and J_Γ . We first compute the K -theory of $C(\blacktriangle)$.

The holes in \blacktriangle are enumerated by elements of \mathbb{F}_3^+ . The central hole corresponds to the empty word \emptyset , while the three holes of the next size down, correspond to the words of length one. In general, for each $w \in \mathbb{F}_3^+$ we enumerate the hole at the centre of $\gamma_w(\blacktriangle)$ by w . The labelling is illustrated in Figure 4.3.

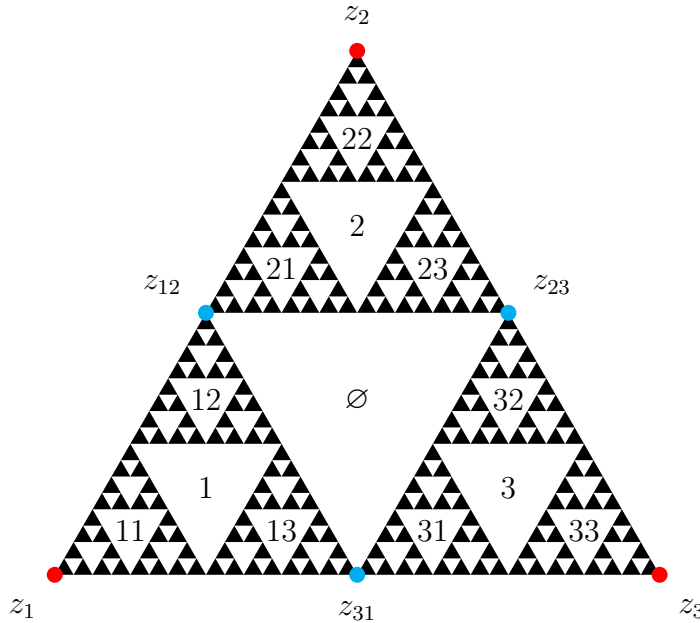


Figure 4.3: The Sierpinski gasket with our hole-labelling convention. The singular boundary $\Delta P_\Gamma = \{z_1, z_2, z_3\}$ is red, while the image of the singular boundary $\Gamma(\Delta P_\Gamma)$ consists of the red and the blue points $\{z_{12}, z_{23}, z_{31}\}$.

Identify \mathbb{R}^2 with \mathbb{C} , and for each $w \in \mathbb{F}_3^+$ let $\lambda_w \in \mathbb{C} \setminus \mathbb{A}$ be the barycentre of the hole enumerated by w . Note that $\gamma_j(\lambda_w) = \lambda_{jw}$. For each $w \in \mathbb{F}_3^+$ fix a unitary $u_w \in C(\mathbb{A})$ with winding number 1 about λ_w and winding number 0 about λ_v for $v \neq w$. The K -theory of $C(\mathbb{A})$ can then be computed using Proposition 4.3.4 and Proposition 4.3.5.

Corollary 4.3.10. *For the Sierpinski gasket \mathbb{A} we have,*

$$K_0(C(\mathbb{A})) = \mathbb{Z}[1_{C(\mathbb{A})}] \quad \text{and} \quad K_1(C(\mathbb{A})) = \bigoplus_{w \in \mathbb{F}_3^+} \mathbb{Z}[u_w].$$

Recall from Example 1.2.15 that the post-critical set P_Γ consists of the 3 corners $z_1 = (0, 0)$, $z_2 = (1/2, \sqrt{3}/2)$, and $z_3 = (1, 0)$. Moreover, $\Delta P_\Gamma = P_\Gamma$. For each $1 \leq j \leq 3$ let $z_j \in \mathbb{A}$ denote the unique element of $\Delta P_\Gamma \cap \gamma_j(\mathbb{A})$. Let $S = \{12, 23, 31\}$ and for each $jk \in S$ we let $z_{jk} \in \mathbb{A}$ denote the unique point contained in $\gamma_j(\mathbb{A}) \cap \gamma_k(\mathbb{A})$, these are highlighted in Figure 4.3. Then $\Delta P_\Gamma = \{z_1, z_2, z_3\}$ and $\Gamma(\Delta P_\Gamma) = \{z_1, z_2, z_3, z_{12}, z_{23}, z_{31}\}$.

We now compute the K -theory of $J_\Gamma = C_0(\mathbb{A} \setminus \{z_1, z_2, z_3\})$ and $I_Y = C_0(\mathbb{A} \setminus \{z_1, z_2, z_3, z_{12}, z_{23}, z_{31}\})$. Let $\iota_{J,A}$ denote the inclusion of J_Γ into $C(\mathbb{A})$ and let $r: C(\mathbb{A}) \rightarrow \Delta P_\Gamma$ be the restriction map. Then the six-term sequence

$$\begin{array}{ccccc} K_0(J_\Gamma) & \xrightarrow{\iota_{J,A*}} & K_0(C(\mathbb{A})) & \xrightarrow{r_*} & K_0(C(\Delta P_\Gamma)) \cong \mathbb{Z}^3 \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(C(\Delta P_\Gamma)) = 0 & \xleftarrow{r_*} & K_1(C(\mathbb{A})) & \xleftarrow{\iota_{J,A*}} & K_1(J_\Gamma) \end{array}$$

is exact. Since $r(1_{C(\mathbb{A})}) = (1, 1, 1) \in \mathbb{C}^3$ and $[1_{C(\mathbb{A})}]$ generates $K_0(C(\mathbb{A}))$, it follows that r_* is injective. Consequently, $\text{im}(\iota_{J,A}) = \{0\}$, and since $K_1(C(\Delta P_\Gamma)) = 0$ it follows that $K_0(J_\Gamma) = 0$. As $K_1(C(\mathbb{A}))$ is free Abelian,

$$K_1(J_\Gamma) \cong K_1(C(\mathbb{A})) \oplus \text{im}(\partial) \cong K_1(C(\mathbb{A})) \oplus \mathbb{Z}^2.$$

Using the exponential map $\partial: K_0(\mathbb{C}^3) \rightarrow K_1(J_\Gamma)$ we can compute explicit generators for $\text{im}(\partial) \subset K_1(J_\Gamma)$. Let \blacktriangle denote the solid triangle in \mathbb{C} with vertices $\{z_1, z_2, z_3\}$. For each $1 \leq j \leq 3$, define a positive function $\tilde{a}_j \in C(\blacktriangle)$ by

$$\tilde{a}_j(z) = \begin{cases} 1 - 2|z - z_j| & \text{if } z \in \gamma_j(\blacktriangle), \\ 0 & \text{otherwise.} \end{cases}$$

Let $a_j \in C(\mathbb{A})$ denote the restriction of \tilde{a}_j to \mathbb{A} , and note that $a_j(z_j) = 1$ and $a_j(z_k) = 0$ for $j \neq k$. Also observe that $\text{supp}(a_j) = \gamma_j(\mathbb{A})$. It follows that a_j is a positive lift of the function $1_{\{z_j\}}$ under $r: C(\mathbb{A}) \rightarrow C(\Delta P_\Gamma)$. Consider the unitary $v_j \in C(\mathbb{A})$ defined by

$$v_j(z) = \exp(2\pi i a_j(z)).$$

For each $w \in \mathbb{F}_3^+$, the function v_j has zero winding number about λ_w . Consequently, $[v_j] = 0$ in $K_1(A)$. As $v_j(z_k) = 1$ for all $1 \leq k \leq 3$, each v_j restricts to a unitary, also denoted v_j , in the minimal unitisation J_Γ^+ . It follows from [RLL00, Proposition 12.2.2] that v_j represents the class $\partial([1_{z_j}])$ in $K_1(J_\Gamma)$. Since $v_j(z_{kl}) = 1$ for all $z_{kl} \in S$, the unitary

v_j also restricts to a unitary in I_Y^+ , again also denoted v_j .

Because $[1_{z_1}] + [1_{z_2}] + [1_{z_3}] = r_*([1_A])$ it follows that $[v_1] + [v_2] + [v_3]$ is the zero class in $K_1(J_\Gamma)$. Therefore, $K_1(J_\Gamma)$ can be described explicitly as

$$K_1(J_\Gamma) \cong \mathbb{Z}[v_1] \oplus \mathbb{Z}[v_2] \oplus \bigoplus_{w \in \mathbb{F}_3^+} \mathbb{Z}[u_w].$$

For I_Y , writing $r: J_\Gamma \rightarrow C(\{z_{12}, z_{23}, z_{31}\})$ for the restriction map, the sequence

$$\begin{array}{ccccc} K_0(I_Y) & \xrightarrow{\iota_{I,J*}} & K_0(J_\Gamma) = 0 & \xrightarrow{r_*} & K_0(C(\partial\Gamma(P_\Gamma))) \cong \mathbb{Z}^3 \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(C(\partial\Gamma(P_\Gamma))) = 0 & \xleftarrow{r_*} & K_1(J_\Gamma) & \xleftarrow{\iota_{I,J*}} & K_1(I_Y) \end{array}$$

is exact. It follows that $K_0(I_Y) = 0$ and $K_1(I_Y) \cong K_1(J_\Gamma) \oplus \mathbb{Z}^3$.

We compute explicit generators for the \mathbb{Z}^3 component. For $jk \in S$ define $b_{jk} \in C(\mathbb{A})$ by

$$b_{jk}(z) = \begin{cases} a_j(\gamma_k^{-1}(z)) & \text{if } z \in \gamma_k(\mathbb{A}), \\ a_k(\gamma_j^{-1}(z)) & \text{if } z \in \gamma_j(\mathbb{A}), \\ 0 & \text{otherwise.} \end{cases} \quad (4.10)$$

Then b_{jk} is a positive function such that $b_{jk}(z_{jk}) = 1$, $b_{jk}(z_{nm}) = 0$ for $nm \neq jk$, and $b_{jk}(z_m) = 0$ for all $1 \leq m \leq 3$. Define unitaries $w_{jk} \in C(\mathbb{A})$ by,

$$w_{jk}(z) = \exp(2\pi i b_{jk}(z)).$$

Identifying w_{jk} with its restriction to I_Y^+ , [RLL00, Proposition 12.2.2] shows that $\partial([1_{z_{jk}}]) = [w_{jk}]$. Consequently,

$$K_1(I_Y) \cong \mathbb{Z}[v_1] \oplus \mathbb{Z}[v_2] \oplus \mathbb{Z}[w_{12}] \oplus \mathbb{Z}[w_{23}] \oplus \mathbb{Z}[w_{31}] \oplus \bigoplus_{w \in \mathbb{F}_3^+} \mathbb{Z}[u_w].$$

Writing $\iota_{I,J*} \in KK(I_Y, J_\Gamma)$ for the KK -class induced by the inclusion, and $\cdot \otimes (\iota_{I,J*} - [Y_\Gamma])$ for the induced map $K_1(I_Y) \rightarrow K_1(J_\Gamma)$, the exact sequence of Corollary 4.3.1 implies that

$$K_0(\mathcal{O}_Y) \cong \ker(\otimes(\iota_{I,J*} - [Y_\Gamma])) \quad \text{and} \quad K_1(\mathcal{O}_Y) \cong \text{coker}(\otimes(\iota_{I,J*} - [Y_\Gamma])).$$

Before we move on to determine the behaviour of the map $\otimes(\iota_{I,J*} - [Y_\Gamma])$, we need to be more explicit with our generators. Currently, the unitaries v_j and w_{jk} both restrict to unitaries in I_Y^+ and J_Γ^+ . The same is not necessarily true of u_w , and so we wish to find unitaries in $C(\mathbb{A})$ with winding number 1 about λ_w , which take the value 1 at each point in $\Gamma(\Delta P_\Gamma)$. We would also like these unitaries to be compatible with the unitaries v_j and w_{jk} to reduce the number of homotopy arguments. To achieve this we introduce yet another family of unitaries.

For each $1 \leq j, k \leq 3$ define $W_{jk} \in C(\blacktriangle)$ by

$$W_{jk}(z) = \begin{cases} \exp(2\pi i a_k(\gamma_j^{-1}(z))) & \text{if } z \in \gamma_j(\blacktriangle), \\ 1 & \text{otherwise.} \end{cases}$$

Then W_{jk} is a unitary in $C(\blacktriangle)$. Note that $W_{jk}W_{kj} = w_{jk}$ for all $jk \in S$. In $C(\blacktriangle)$ the unitaries W_{jk} with $j \neq k$ have winding number about λ_\emptyset equal to 1 if $jk \in S$ and -1 otherwise. They also have winding number 0 about λ_w for $w \neq \emptyset$. Thus, in $K_1(C(\blacktriangle))$ we have,

$$[W_{12}] = [W_{31}] = [W_{23}] = [u_\emptyset] = -[W_{21}] = -[W_{13}] = -[W_{31}].$$

Unlike the unitary u_\emptyset , the unitary W_{jk} takes the value 1 on each point in $\Gamma(\Delta P_\Gamma)$. Therefore, each W_{jk} restricts to a elements of both I_Y^+ and J_Γ^+ . It follows that the maps on K -theory induced by the inclusions of I_Y into $C(\blacktriangle)$ and J_Γ into $C(\blacktriangle)$, both take $[W_{jk}]$ to $[u_\emptyset]$ for all $jk \in S$.

The classes $[W_{jk}]$ are mutually distinct in both $K_1(I_Y)$ and $K_1(J_\Gamma)$ for each $jk \in S$. In Proposition 4.3.13 we show that $[W_{jk}]$ can be written as a linear combination of terms involving the one of the $[W_{nm}]$ classes, together with the classes $[w_{jk}]$ and $[v_j]$.

Since $W_{jk}W_{kj} = w_{jk}$, in both $K_1(I_Y)$ and $K_1(J_\Gamma)$ we have

$$[W_{jk}] + [W_{kj}] = [w_{jk}] \quad (4.11)$$

for all $jk \in S$. However, $[w_{ij}] = 0$ in $K_1(J_\Gamma)$, as $[w_{ij}] \in K_1(I_Y)$ is in the image of $\partial: K_1(C(\partial\Gamma(P_\Gamma))) \rightarrow K_1(I_Y)$. The classes $[W_{ij}]$ also satisfy a second identity; to see this we require the following lemma.

Lemma 4.3.11. *For $1 \leq j \leq 3$ we have $[v_j] = [W_{jj}]$ in each of $K_1(C(\blacktriangle))$, $K_1(J_\Gamma)$, and $K_1(I_Y)$.*

Proof. We prove the result when $j = 1$; the rest follow by symmetry. Recall that $z_1 = 0$. For each $t \in [1, 2]$ define $\tilde{a}_{1,t}$ in $C(\blacktriangle)$ by

$$\tilde{a}_{1,t}(z) = \begin{cases} 1 - 2|tz - z_1| & \text{if } tz \in \gamma_1(\blacktriangle), \\ 0 & \text{otherwise.} \end{cases}$$

Let $a_{1,t}$ be the restriction of $\tilde{a}_{1,t}$ to \blacktriangle . Then $t \mapsto \exp(2\pi i a_{1,t})$ is a continuous path of unitaries in $C(\blacktriangle)$ from v_1 to W_{11} . For all $t \in [1, 2]$, the value of $a_{1,t}$ each point of $\Gamma(\Delta P_\Gamma)$ is 1. In particular, the homotopy restricts to homotopies of unitaries in both J_Γ^+ and I_Y^+ . \square

Lemma 4.3.12. *For each $1 \leq j \leq 3$, we have*

$$[W_{j1}] + [W_{j2}] + [W_{j3}] = 0 \quad (4.12)$$

in both $K_1(J_\Gamma)$ and $K_1(I_Y)$.

Proof. The one point compactification of $\blacktriangle \setminus \Gamma(\Delta P_\Gamma)$ is homeomorphic to the wedge sum of three copies of the one point compactification of $\blacktriangle \setminus \Delta P_\Gamma$ with common basepoint

∞ . Each copy contains precisely one of the sets $\gamma_j(\mathbb{A} \setminus \Delta P_\Gamma)$, and we denote the copy containing $\gamma_j(\mathbb{A} \setminus \Delta P_\Gamma)$ by Z_j . The key observation is that W_{jk} “looks like” a copy of v_k in Z_j , and $\sum_{k=1}^3 [v_k] = 0$.

To be more precise, let $R: I_Y \rightarrow C(Z_j)$ denote restriction of functions. Upon identifying $C(Z_j)$ with J_Γ^+ we see that $R_*([W_{jk}]) = [v_k]$. Since $\sum_{k=1}^3 [v_k] = 0$, it follows that $\sum_{k=1}^3 [W_{jk}] \in \ker(R_*)$. The restriction map R splits via the $*$ -homomorphism $\alpha: C(Z_j) \rightarrow I_Y$ defined by

$$\alpha(f)(z) = \begin{cases} f(z) & \text{if } z \in Z_j, \\ f(\infty) & \text{otherwise.} \end{cases}$$

Consequently, $\sum_{k=1}^3 [W_{jk}] = 0$ in $K_1(I_Y)$. Thus, $\sum_{k=1}^3 \iota_{I,J_*}([W_{jk}]) = 0$ in $K_1(J_\Gamma)$. \square

We obtain the following relation between elements of $K_1(I_Y)$ and $K_1(J_\Gamma)$.

Proposition 4.3.13. *For each $jk \in S$ and $i \neq j, k$ we have*

$$[W_{ij}] = [v_j] + [W_{jk}]$$

in $K_1(J_\Gamma)$ and

$$[W_{ij}] = [w_{ij}] + [v_j] + [W_{jk}]$$

in $K_1(I_Y)$.

Proof. Using (4.11), (4.12), and Lemma 4.3.11 we have

$$[W_{ij}] = [w_{ij}] - [W_{ji}] = [w_{ij}] + [W_{jj}] + [W_{jk}] = [w_{ij}] + [v_j] + [W_{jk}].$$

Since $[w_{ij}] = 0$ in $K_1(J_\Gamma)$ the first identity also holds. \square

In light of Proposition 4.3.13 we can choose any one of the unitaries W_{12} , W_{23} , or W_{31} to represent a class in $K_1(I_Y)$ and $K_1(J_\Gamma)$ which gets sent to $[u_\emptyset]$ via the map on K -theory induced by inclusion into $C(\mathbb{A})$. We choose W_{12} .

For each $w \in \mathbb{F}_3^+ \setminus \{\emptyset\}$, define unitaries $U_w \in C(\mathbb{A})$ by

$$U_w(z) = \begin{cases} W_{12}(\gamma_w^{-1}(z)) & \text{if } z \in \gamma_w(z); \\ 1 & \text{otherwise.} \end{cases}$$

For convenience, we also let $U_\emptyset = W_{12}$. Then U_w takes the value 1 at each point of $\Gamma(\Delta P_\Gamma)$, and accordingly restricts to a unitary in both I_Y^+ and J_Γ^+ . For each $w \in \mathbb{F}_3^+$, U_w has winding number 1 about λ_w and winding number 0 about λ_v for $v \neq w$. Hence, $[U_w] = [u_w]$ in $K_1(C(\mathbb{A}))$.

The K_1 -groups are now given with explicit generators by

$$K_1(J_\Gamma) = \mathbb{Z}[v_1] \oplus \mathbb{Z}[v_2] \oplus \bigoplus_{w \in \mathbb{F}_3^+ \setminus \{\emptyset\}} \mathbb{Z}[U_w], \quad \text{and}$$

$$K_1(I_Y) = \mathbb{Z}[v_1] \oplus \mathbb{Z}[v_2] \oplus \mathbb{Z}[w_{12}] \oplus \mathbb{Z}[w_{23}] \oplus \mathbb{Z}[w_{31}] \oplus \bigoplus_{w \in \mathbb{F}_3^+ \setminus \{\emptyset\}} \mathbb{Z}[U_w].$$

We now return to describing the map $\otimes(\iota_{I,J*} - [Y_\Gamma]): K_1(I_Y) \rightarrow K_1(J_\Gamma)$. For $\iota_{I,J*}$, our choice of representatives implies that

$$\iota_{I,J*}([v_j]) = [v_j], \quad \iota_{I,J*}([w_{jk}]) = 0, \quad \text{and} \quad \iota_{I,J*}([U_w]) = [U_w].$$

The problem now falls to describing the Kasparov product $\cdot \otimes [Y_\Gamma]$, for which we rely on Proposition 4.3.3. In particular, $\cdot \otimes [Y_\Gamma] = \sum_{\gamma \in \Gamma} (\alpha_\gamma \circ r_\gamma)_*$. Starting with $[v_j]$, we use Lemma 4.3.11 to see that $[v_j] = [W_{jj}]$ in $K_1(I_Y)$. Since $\text{supp}(a_j) \subseteq \gamma_j(\Delta)$, we have

$$[v_j] \otimes [Y_\Gamma] = (\alpha_{\gamma_j} \circ r_{\gamma_j})_*([v_j]) = (\alpha_{\gamma_j} \circ r_{\gamma_j})_*([W_{jj}]).$$

Recall that α_{γ_j} is the $*$ -homomorphism dual to $\gamma_j: C_0(\Delta \setminus \Delta P_\Gamma) \rightarrow C_0(\Delta \setminus \Gamma(\Delta P_\Gamma))$. The construction of W_{jj} implies that $(\alpha_{\gamma_j} \circ r_{\gamma_j})^+: I_Y^+ \rightarrow J_Y^+$ takes W_{jj} to v_j . It now follows that $[v_j] \otimes [Y_\Gamma] = [v_j]$.

Now consider $[w_{jk}]$. Recall that the unitaries w_{jk} are defined by exponentiating the functions b_{jk} of (4.10). Since $\text{supp}(b_{jk}) \subseteq \gamma_j(\Delta) \cup \gamma_k(\Delta)$, Proposition 4.3.3 implies that

$$[w_{jk}] \otimes [Y_\Gamma] = (\alpha_{\gamma_j} \circ r_{\gamma_j})_*([w_{jk}]) + (\alpha_{\gamma_k} \circ r_{\gamma_k})_*([w_{jk}]).$$

It follows from the definition of b_{jk} that

$$(\alpha_{\gamma_j} \circ r_{\gamma_j})_*[w_{jk}] = (\alpha_{\gamma_j})_*([W_{jk}]) = [v_k]$$

and

$$(\alpha_{\gamma_k} \circ r_{\gamma_k})_*[w_{jk}] = (\alpha_{\gamma_k})_*([W_{kj}]) = [v_j].$$

Now consider $[U_w]$ for $w \in \mathbb{F}_3^+$. First suppose that $w = w_1 \cdots w_n \neq \emptyset$. By construction of the U_w , we have $\gamma_{w_1}^*(U_{w_1 w_2 \cdots w_n}) = U_{w_2 \cdots w_n}$ in $C(\Delta)$. Since U_w is identically 1 outside $\gamma_{w_1}(\Delta)$, Proposition 4.3.3 gives

$$[U_{w_1 \cdots w_n}] \otimes [Y_\Gamma] = (\alpha_{\gamma_{w_1}} \circ r_{\gamma_{w_1}})_*([U_{w_1 \cdots w_n}]) = \begin{cases} [U_{w_2 \cdots w_n}] & \text{if } \ell(w) > 1, \\ [U_\emptyset] & \text{if } \ell(w) = 1. \end{cases}$$

Finally, consider the class $[U_\emptyset]$. Observe that since $U_\emptyset = W_{12}$, we have $\gamma_1^*(U_\emptyset) = v_2$ in $C(\Delta)$. Since U_\emptyset is identically 1 outside $\gamma_1(\Delta)$, it follows from Proposition 4.3.3 that

$$[U_\emptyset] \otimes [Y_\Gamma] = (\alpha_{\gamma_1} \circ r_{\gamma_1})_*([U_\emptyset]) = [v_2]. \quad (4.13)$$

Remark 4.3.14. The apparent asymmetry induced in $[U_\emptyset] \in K_1(I_Y)$ being mapped to $[v_2] \in K_1(J_\Gamma)$ is not an issue. For example, if we had instead used $[W_{23}] \in K_1(I_Y)$ to define U_\emptyset , we would have obtained $[W_{23}] \otimes [Y_\Gamma] = [v_3]$ by an argument similar to (4.13), and then Proposition 4.3.13 would have given

$$\begin{aligned} [W_{12}] \otimes [Y_\Gamma] &= [w_{12}] \otimes [Y_\Gamma] + [v_2] \otimes [Y_\Gamma] + [W_{23}] \otimes [Y_\Gamma] \\ &= [v_1] + [v_2] + [v_2] + [v_3] \\ &= [v_2]. \end{aligned}$$

A similar calculation could be repeated with $[W_{31}]$.

To summarise, the map $\cdot \otimes (\iota_{I,J*} - [Y_\Gamma]): K_1(I_Y) \rightarrow K_1(J_\Gamma)$ is given by

$$\begin{aligned} [v_j] &\mapsto 0 \\ [w_{jk}] &\mapsto -[v_j] - [v_k] = [v_l] & (l \neq j, k) \\ [U_{w_1 w_2 \dots w_n}] &\mapsto [U_{w_1 w_2 \dots w_n}] - [U_{w_2 \dots w_n}] & (\ell(w) > 1) \\ [U_{w_1}] &\mapsto [U_w] - [U_\emptyset] & (\ell(w) = 1) \\ [U_\emptyset] &\mapsto [U_\emptyset] - [v_2]. \end{aligned}$$

With this we can compute the K -groups of \mathcal{O}_Y . For K_0 we have

$$K_0(\mathcal{O}_Y) \cong \ker(\cdot \otimes (\iota_{I,J*} - [Y_\Gamma])) = \mathbb{Z}[w_{12}w_{23}w_{13}] \oplus \mathbb{Z}[v_1] \oplus \mathbb{Z}[v_2].$$

For each $w = w_1 \dots w_n \in \mathbb{F}_3^+$, we have the telescoping series

$$[U_w] = ([U_w] - [U_{w_2 \dots w_n}]) + ([U_{w_2 \dots w_n}] - [U_{w_3 \dots w_n}]) + \dots ([U_{w_n}] - [U_\emptyset]) + ([U_\emptyset] - [v_2]) + [v_2].$$

It follows that $[U_w]$ is in the image of $\cdot \otimes (\iota_{I,J*} - [Y_\Gamma])$. Since each $[v_j]$ is in the image of $\cdot \otimes (\iota_{I,J*} - [Y_\Gamma])$, it follows that $\cdot \otimes (\iota_{I,J*} - [Y_\Gamma])$ is surjective and

$$K_1(\mathcal{O}_Y) \cong \operatorname{coker}(\cdot \otimes (\iota_{I,J*} - Y_\Gamma)) = 0.$$

Since (\mathbb{A}, Γ) is contractive and has no branched points, Proposition 2.3.2 implies that the associated Kajiwara-Watatani algebra $C^*(\mathbb{A}, \Gamma)$ is isomorphic to \mathcal{O}_3 . So its K -groups are $K_0(C^*(\mathbb{A}, \Gamma)) \cong \mathbb{Z}/2\mathbb{Z}$ and $K_1(C^*(\mathbb{A}, \Gamma)) = 0$. The K_0 -group for \mathcal{O}_Y is notably different, since it is torsion free.

The generators of $K_0(\mathcal{O}_Y) \cong \ker(\cdot \otimes (\iota_{I,J*} - [Y_\Gamma]))$ can also be seen to encode information about the interaction of topology and dynamics. In particular, the class $[w_{12}w_{23}w_{13}]$ belongs to the kernel of $\cdot \otimes (\iota_{I,J*} - [Y_\Gamma])$, since it is sent via $\cdot \otimes [Y_\Gamma]$ (the dynamics) to $[v_1] + [v_2] + [v_3]$. Then $[v_1] + [v_2] + [v_3] = 0$ due to the topology of $\mathbb{A} \setminus \{z_1, z_2, z_3\}$.

A similar analysis could also be undertaken for the twisted gasket system $(\mathbb{A}, \Gamma' = \{\gamma'_1, \gamma_2, \gamma_3\})$ of Section 2.4.2, or for that matter any other iterated function system with attractor \mathbb{A} .

4.3.3 Square dynamics

There are numerous possibilities for iterated function systems on the square $\square := [0, 1] \times [0, 1]$, but we restrict ourselves to Example 4.1.10 and Example 4.1.11. Both of these examples have singular boundary equal to the perimeter of \square . Consequently, $\square \setminus \Delta P_\Gamma = (0, 1) \times (0, 1)$ and $J_\Gamma \cong C_0(\mathbb{R}^2)$. Therefore, the associated lacunary correspondences have the same coefficient algebra. Noting that $C_0(\mathbb{R}^2)^+ \cong C(S^2)$ let $p_{Bott} \in M_2(C(S^2))$ denote the Bott projection (see [GBVF01, §2.6]). Then

$$K_0(J_\Gamma) = \mathbb{Z}([p_{Bott}] - [1]) \quad \text{and} \quad K_1(J_\Gamma) = 0.$$

Example 4.3.15. Let $(\square, \Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\})$ denote the iterated function system from Example 4.1.10. From Figure 4.1 we see that the set $\Gamma(\Delta P_\Gamma)$ consists of the union of the perimeters of the squares $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$, and \mathbb{A}_4 . Hence, the set $\square \setminus \Gamma(\Delta P_\Gamma)$ is homeomorphic to the disjoint union of 4 copies of \mathbb{R}^2 . Letting $p_{Bott,i} \in M_2(C(S^2))$ denote the Bott projection coming from the one-point compactification of $\mathbb{A}_i \setminus \Gamma(\Delta P_\Gamma) \simeq \mathbb{R}^2$, we have

$$K_0(I_Y) = \bigoplus_{i=1}^4 \mathbb{Z}([p_{Bott,i}] - [1]) \quad \text{and} \quad K_1(I_Y) = 0.$$

Writing $\iota_{I,J*} \in KK(I_Y, J_\Gamma)$ for the KK -class induced by the inclusion, and $\cdot \otimes (\iota_{I,J*} - [Y_\Gamma])$ for the induced map $K_1(I_Y) \rightarrow K_1(J_\Gamma)$, the exact sequence of Corollary 4.3.1 implies that

$$K_0(\mathcal{O}_Y) \cong \ker(\otimes(\iota_{I,J*} - [Y_\Gamma])) \quad \text{and} \quad K_1(\mathcal{O}_Y) \cong \text{coker}(\otimes(\iota_{I,J*} - [Y_\Gamma])).$$

Once again we need to understand the maps $\iota_{I,J*}$ and $\cdot \otimes [Y_\Gamma]$.

Lemma 4.3.16. For each $1 \leq i \leq 4$ we have $\iota_{I,J*}([p_{Bott,i}] - [1]) = [p_{Bott}] - [1]$.

Proof. We show that the result holds for $i = 1$. The rest follow from symmetry. Consider the Bott projection $p_{Bott} \in M_2(C_0((0,1) \times (0,1))^+)$. Without loss of generality we can assume that $p_{Bott}(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. For each $t \in [1/2, 1]$ consider the projection $p_t \in M_2(C_0((0,1) \times (0,1))^+)$ given by

$$p_t(x, y) = \begin{cases} p_{Bott}(x/t, y/t) & \text{if } (x, y) \in (0, t) \times (0, t); \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{otherwise (this includes } (x, y) = \infty \text{).} \end{cases}$$

Then $t \mapsto p_t$ defines a homotopy between p_{Bott} and $\iota_{I,J}(p_{Bott,1})$. The result now follows. \square

For the map $\cdot \otimes [Y_\Gamma]$ we use Proposition 4.3.3. Considering $p_{Bott,i}$ as an element of $M_2(C_0(\square \setminus \Gamma(\Delta P_\Gamma))^+)$, we see that $p_{Bott,i}(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for all $(x, y) \in \mathbb{A}_j \setminus \Gamma(\Delta P_\Gamma)$ with $j \neq i$. Consequently,

$$([p_{Bott,i}] - [1]) \otimes [Y_\Gamma] = (\alpha_{\gamma_i} \circ r_{\gamma_i})_*([p_{Bott,i}] - [1]) = [p_{Bott}] - [1].$$

It follows from Lemma 4.3.16 that $\cdot \otimes (\iota_{I,J*} - [Y_\Gamma]): K_0(I_Y) \rightarrow K_0(J_\Gamma)$ is the zero map. Hence,

$$K_1(\mathcal{O}_Y) \cong K_0(I_Y) = \bigoplus_{i=1}^4 \mathbb{Z}([p_{Bott,i}] - [1]) \quad \text{and} \quad K_0(\mathcal{O}_Y) \cong K_0(J_\Gamma) = \mathbb{Z}([p_{Bott}] - [1]).$$

The relatively simple nature of the K -groups in this example is unsurprising, since the overlap structure in (\square, Γ) is fairly boring. We remark that since (\square, Γ) is contractive and has no branched points, the Kajiwara-Watatani algebra $C^*(\square, \Gamma)$ is isomorphic to the \mathcal{O}_{X_Γ} which is in turn isomorphic to the Cuntz algebra \mathcal{O}_4 by Proposition 2.1.4. Once again, the K -groups we have computed are distinct from the K -groups of the Cuntz algebra. Indeed, both $K_0(\mathcal{O}_Y)$ and $K_1(\mathcal{O}_Y)$ are non-trivial and torsion free. \triangle

Example 4.3.17. Let $(\square, \Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\})$ denote the iterated function system from Example 4.1.11. Recall from Figure 4.2 that $\square \setminus \Gamma(\Delta P_\Gamma)$ consists of 9 open squares. In particular, $\square \setminus \Gamma(\Delta P_\Gamma)$ is homeomorphic to the disjoint union of 9 copies of \mathbb{R}^2 . We denote the squares as follows:

- for $1 \leq i \leq 4$ let S_i denote the unique open square such that $S_i \cap \mathbb{A}_j = \emptyset$ for all $j \neq i$;
- let $L = \{12, 13, 24, 34\}$ and for each $ij \in L$ let S_{ij} denote the unique open square contained in $\mathbb{A}_i \cap \mathbb{A}_j$ satisfying $S_{ij} \cap \mathbb{A}_k = \emptyset$ for $k \neq i, j$.
- let S_{1234} denote the unique open square contained in $\bigcap_{i=1}^4 \mathbb{A}_i$.

For convenience we write $M = \{1, 2, 3, 4, 12, 13, 24, 34, 1234\}$. The labelling can be seen in Figure 4.4.

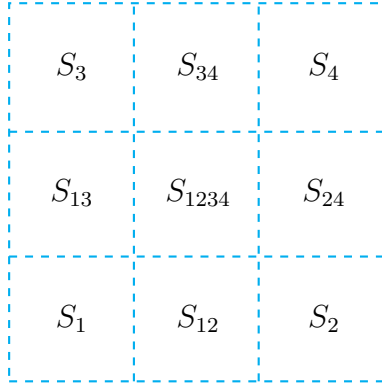


Figure 4.4: The labelling of the open squares in $\square \setminus \Gamma(\Delta P_\Gamma)$ for the iterated function system of Example 4.3.17.

For each $w \in M$ we let $p_{Bott,w}$ denote the Bott projection in $M_2(S_w^+)$. Then

$$K_0(I_Y) = \bigoplus_{w \in M} \mathbb{Z}([p_{Bott,w}] - [1]) \quad \text{and} \quad K_1(I_Y) = 0.$$

Once again, with $\cdot \otimes (\iota_{I,J*} - [Y_\Gamma]): K_0(I_Y) \rightarrow K_0(J_\Gamma)$, the exact sequence of Corollary 4.3.1 implies that

$$K_1(\mathcal{O}_Y) \cong \ker(\otimes(\iota_{I,J*} - [Y_\Gamma])) \quad \text{and} \quad K_0(\mathcal{O}_Y) \cong \text{coker}(\otimes(\iota_{I,J*} - [Y_\Gamma])).$$

A homotopy argument similar to that found in the proof of Lemma 4.3.16 gives

$$\iota_{I,J*}([p_{Bott,w}] - [1]) = [p_{Bott}] - [1]$$

for all $w \in M$. Considering $p_{Bott,w}$ as an element of $M_2(C_0(S_w)^+)$, we see that $p_{Bott,w}(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ whenever (x, y) belongs to the complement of S_w in $\square \setminus \Gamma(\Delta P_\Gamma)$. Consequently,

$$\begin{aligned}
& ([p_{Bott,w}] - [1]) \otimes [Y_\Gamma] \\
&= \begin{cases} (\alpha_{\gamma_w} \circ r_{\gamma_w})_*([p_{Bott,w}] - [1]) & \text{if } w \in \{1, 2, 3, 4\}, \\ (\alpha_{\gamma_i} \circ r_{\gamma_i})_*([p_{Bott,w}] - [1]) + (\alpha_{\gamma_{ij}} \circ r_{\gamma_j})_*([p_{Bott,w}] - [1]) & \text{if } w = ij \in L, \\ \sum_{i=1}^4 (\alpha_{\gamma_i} \circ r_{\gamma_i})_*([p_{Bott,w}] - [1]) & \text{if } w = 1234, \end{cases} \\
&= \begin{cases} [p_{Bott}] - [1] & \text{if } w \in \{1, 2, 3, 4\}, \\ 2([p_{Bott}] - [1]) & \text{if } w = ij \in L, \\ 4([p_{Bott}] - [1]) & \text{if } w = 1234. \end{cases}
\end{aligned}$$

To shorten notion we let x_w denote the class $[p_{Bott,w}] - [1] \in K_0(I_Y)$. Then,

$$x_w \otimes (\iota_{I,J*} - [Y_\Gamma]) = \begin{cases} 0 & \text{if } w \in \{1, 2, 3, 4\}, \\ -([p_{Bott}] - [1]) & \text{if } w = ij \in L, \\ -3([p_{Bott}] - [1]) & \text{if } w = 1234. \end{cases}$$

Since $\cdot \otimes (\iota_{I,J*} - [Y_\Gamma])$ is surjective

$$K_0(\mathcal{O}_Y) = 0$$

Also,

$$\begin{aligned}
K_1(\mathcal{O}_Y) &\cong \ker(\otimes(\iota_{I,J*} - [Y_\Gamma])) \\
&= \mathbb{Z}(x_{12} - x_{24}) \oplus \mathbb{Z}(x_{24} - x_{34}) \oplus \mathbb{Z}(x_{34} - x_{13}) \\
&\quad \oplus \mathbb{Z}(x_{1234} - x_{12} - x_{24} - x_{34}) \oplus \bigoplus_{i=1}^4 \mathbb{Z}x_i \\
&\cong \mathbb{Z}^8.
\end{aligned}$$

Once again the Kajiwara-Watatani algebra $C^*(\square, \Gamma)$ of this system is isomorphic to \mathcal{O}_4 . \(\triangle\)

Although the iterated function systems from Example 4.3.15 and Example 4.3.17 contain the same number of maps, and the associated correspondences have the same coefficient algebra $J_\Gamma = C_0((0, 1) \times (0, 1))$, we obtain strikingly different K -groups in each case. The K -groups of Example 4.3.17 clearly have something to do with the overlap structure of the underlying iterated function system.

Investigating the lacunary algebra warrants further analysis. Determining precisely which features of the interaction between the topology and dynamics of an iterated function system are captured by this algebra will be the subject of future work.

4.4 | Critical points and inner products

To finish this chapter, we move away from the lacunary algebra and instead look at the relationship between critical points and inner products on Hilbert modules. We require some topological prerequisites.

Definition 4.4.1 (cf. [HP97, Definition 2.3]). Let X and Y be topological spaces. A function $F: X \rightarrow 2^Y \setminus \{\emptyset\}$ is said to be *lower semi-continuous at $x \in X$* or *lower Vietoris continuous at $x \in X$* if for every open set $V \subseteq Y$ such that $V \cap F(x) \neq \emptyset$, there exists a neighbourhood U of x such that $F(x') \cap V \neq \emptyset$ for all $x' \in U$. We say that F is *lower semi-continuous* or *lower Vietoris continuous* if it is lower semi-continuous for all $x \in X$.

The following is a local version of [PT11, Definition 2.7].

Definition 4.4.2. Let X and Y be topological spaces. A map $p: Y \rightarrow X$ is said to be *locally surjective at $y \in Y$* if for any open neighbourhood U of y , there is another open neighbourhood $U_y \subseteq U$ of y such that $p(U_y)$ is an open neighbourhood of $p(y)$.

Lemma 4.4.3 ([PT11, Lemma 2.8]). *Let X and Y be topological spaces. Then a map $p: Y \rightarrow X$ is open if and only if it is locally surjective at each $y \in Y$.*

The following result is a local version of [Kie02, Proposition 1.4.16 (iii)], which relates lower semi-continuity to local surjectivity.

Proposition 4.4.4. *Let X and Y be compact Hausdorff spaces, and suppose that $p: Y \rightarrow X$ is a continuous surjection. Then $p^{-1}: X \rightarrow 2^Y \setminus \{\emptyset\}$ is lower semi-continuous at $x \in X$ if and only if for all $y \in p^{-1}(x)$ the function p is locally surjective at y . Moreover, p^{-1} is lower semi-continuous if and only if p is open.*

Proof. Fix $x \in X$. Take an open set V in Y such that $p^{-1}(x) \cap V \neq \emptyset$, and suppose that for each $y \in p^{-1}(x) \cap V$ the map p is locally surjective at y . Then there is an open neighbourhood $V_y \subseteq V$ of y such that $p(V_y)$ is an open neighbourhood of $p(y) = x$. Let $U = \bigcup_{y \in p^{-1}(x) \cap V} p(V_y)$. Then U is an open neighbourhood of x and for each $x' \in U$, the set $p^{-1}(x') \cap V$ is non-empty. Thus, p^{-1} is lower semi-continuous at x .

Now suppose that p^{-1} is lower semi-continuous at x . Fix $y \in p^{-1}(x)$. Then there is an open neighbourhood U of x such that $p^{-1}(x') \cap V \neq \emptyset$ for all $x' \in U$. Let $V_y = p^{-1}(U) \cap V$. We claim that $p(V_y) = U$. Clearly, if $x' \in p(p^{-1}(U) \cap V)$ then $x' \in U$. For the reverse inclusion suppose that $x' \in U$. Then $p^{-1}(x') \cap V \neq \emptyset$. Taking $y' \in p^{-1}(x') \cap V$ gives $x' = p(y') \in p(V_y)$. Consequently, p is locally surjective at y .

The final statement now follows from Lemma 4.4.3 □

Suppose that $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ is an injective iterated function system. Recall the correspondence $X_\Gamma = C(\mathbb{A} \times \Gamma)$ over $C(\mathbb{A})$ from Section 2.1. Let $p_1: \mathbb{A} \times \Gamma \rightarrow \mathbb{A}$ denote the map $p_1(x, \gamma) = \gamma(x)$. Then p_1 is a continuous surjection. Using Proposition 4.4.4 we will describe the obstruction to the map p_1 being open in terms of the critical set of (\mathbb{A}, Γ) .

Definition 4.4.5. Let $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ be an injective iterated function system. Define the *critical boundary* to be the set

$$\Delta C_\Gamma := \bigcup_{i=1}^N \partial \mathbb{A}_i.$$

By definition, the critical boundary is a subset of $\Gamma(\Delta P_\Gamma)$. Note that ΔC_Γ is not necessarily the topological boundary of the critical set C_Γ , but the following is true.

Lemma 4.4.6. *Let (\mathbb{A}, Γ) be an injective iterated function system. Then $\Delta C_\Gamma \subseteq C_\Gamma$.*

Proof. Let $x \in \Delta C_\Gamma$. Then there exists $\gamma \in \Gamma$ such that $x \in \partial\gamma(\mathbb{A})$. Fix a sequence $x_n \rightarrow x$ with $x_n \in \mathbb{A} \setminus \gamma(\mathbb{A})$. Since $\mathbb{A} = \bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A})$, by passing to a subsequence if necessary, we can assume there exists $\gamma' \neq \gamma$ such that $x_n \in \gamma'(\mathbb{A})$. Since $\gamma'(\mathbb{A})$ is closed, it follows that $x \in \gamma'(\mathbb{A})$. Hence $x \in C_\Gamma$. \square

The critical boundary is the obstruction to lower semi-continuity of p_1^{-1} , and by Proposition 4.4.4, also the obstruction to openness of p_1 .

Lemma 4.4.7. *Let $(\mathbb{A}, \Gamma = \{\gamma_1, \dots, \gamma_N\})$ be an injective iterated function system. Then $p_1^{-1}: \mathbb{A} \rightarrow 2^{\mathbb{A} \times \Gamma} \setminus \{\emptyset\}$ is lower semi-continuous at x if and only if $x \in \mathbb{A} \setminus \Delta C_\Gamma$.*

Proof. First suppose that $x \in \Delta C_\Gamma$. Then there exists $1 \leq i \leq N$ such that $x \in \partial\mathbb{A}_i$. Let U be an open neighbourhood of x . Then $U \setminus \mathbb{A}_i \neq \emptyset$. In particular, there exists $x' \in U \setminus \mathbb{A}_i$. Now, the set $\mathbb{A} \times \{\gamma_i\}$ is open in $\mathbb{A} \times \Gamma$ and $p_1^{-1}(x) \cap (\mathbb{A} \times \{\gamma_i\}) \neq \emptyset$. However, $p_1^{-1}(x') \cap (\mathbb{A} \times \{\gamma_i\}) = \emptyset$ since $x' \notin \mathbb{A}_i$. Thus, p_1^{-1} is not lower semi-continuous at x .

Conversely, suppose that p_1^{-1} is not lower semi-continuous at $x \in \mathbb{A}$. Then there exists an open subset $V \subseteq \mathbb{A} \times \Gamma$ with $V \cap p_1^{-1}(x) \neq \emptyset$ such that every open neighbourhood U of x contains some $x' \in U$ with $p_1^{-1}(x') \cap V = \emptyset$. Consequently, we can find a sequence $x_n \rightarrow x$ such that $p_1^{-1}(x_n) \cap V = \emptyset$ for all $n \in \mathbb{N}$.

Since $V \cap p_1^{-1}(x) \neq \emptyset$, there exists $1 \leq i \leq N$ with $x \in \mathbb{A}_i$ such that $(\gamma_i^{-1}(x), \gamma_i) \in V$. We claim that $\{n \mid x_n \in \mathbb{A}_i\}$ is finite. Suppose for contradiction that there exists a subsequence such that $x_n \in \mathbb{A}_i$ for all $n \in \mathbb{N}$. Let $y_n = \gamma_i^{-1}(x_n)$. By compactness of \mathbb{A} there is a subsequence such that $y_n \rightarrow y$ for some $y \in \mathbb{A}$. Since γ_i is continuous, $x_n = \gamma_i(y_n) \rightarrow \gamma_i(y)$, and so $\gamma_i(y) = x$. Hence

$$(\gamma_i^{-1}(x_n), \gamma_i) = (y_n, \gamma_i) \rightarrow (y, \gamma_i) = (\gamma_i^{-1}(x), \gamma_i).$$

This contradicts $p_1^{-1}(x_n) \cap V = \emptyset$. Thus, $\{n \mid x_n \in \mathbb{A}_i\}$ is finite. By passing to a subsequence, we can now assume that $x_n \notin \mathbb{A}_i$ for all $n \in \mathbb{N}$. Since $x_n \rightarrow x \in \mathbb{A}_i$, we conclude that $x \in \partial\mathbb{A}_i \subseteq \Delta C_\Gamma$. \square

We now relate critical boundary to inner products. In [PT11, Theorem 4.3] it is claimed that if $p: Y \rightarrow X$ is a continuous surjection, then $C(Y)$ can be equipped with a $C(X)$ -valued inner product in such a way that $C(Y)$ becomes a $C(X)$ -module with a norm equivalent to the supreme norm on $C(Y)$. We prove a converse of sorts to this result.

Proposition 4.4.8. *Let $p: Y \rightarrow X$ be a continuous surjection between compact Hausdorff spaces. Let $A = C(X)$ and define a right A -action on $C(Y)$ by $(\xi \cdot a)(y) = \xi(y)a(p(y))$ for all $\xi \in C(Y)$ and $a \in A$. If p is not open, then there is no A -valued right inner product on $C(Y)$ that is compatible with the right A -action and induces a norm equivalent to the supremum norm on $C(Y)$.*

Proof. Suppose $(\cdot \mid \cdot)_A$ is an inner product on $C(Y)$ that is compatible with the right A -module structure. Since p is not open, Proposition 4.4.4 implies that there exist $x \in X$, $y \in p^{-1}(x)$, and an open neighbourhood V of y , such that for any open neighbourhood U

of x , there exists $x' \in U$ with $p^{-1}(x') \cap V = \emptyset$. Let $\mathcal{B}(x)$ be a neighbourhood basis of x . Then for any $U \in \mathcal{B}(x)$ the set $U \setminus p(V)$ is non-empty.

Since Y is normal, we can find an open neighbourhood W of y such that $\overline{W} \subseteq V$. Since Y is compact, the map p is closed, so $p(\overline{W})$ is closed in X . Moreover, $U \setminus p(\overline{W}) \subseteq U \setminus p(V)$ for any $U \in \mathcal{B}(x)$. For each $U \in \mathcal{B}(x)$ let $x_U \in U \setminus p(\overline{W})$. Then $(x_U)_{U \in \mathcal{B}(x)}$ is a net that converges to x .

Using the Tietze Extension Theorem take, for each $U \in \mathcal{B}(x)$, a positive function $e_U \in A$ such that $e_U(x_U) = 1$ and $e_U(x') = 0$ for all $x' \in p(\overline{W})$. Also, take $\xi \in C(Y)$ such that $\xi(y) = 1$ and $\xi(y') = 0$ for all $y' \in Y \setminus W$. Because of the chosen supports, we have $\xi \cdot e_U = 0$ for all $U \in \mathcal{B}(x)$. It now follows that,

$$(\xi \mid \xi)_A(x_U) = (\xi \mid \xi)_A(x_U)e_U(x_U) = (\xi \mid \xi \cdot e_U)(x_U) = 0.$$

Continuity of $(\xi \mid \xi)_A$ forces $(\xi \mid \xi)_A(x) = 0$. By the Tietze Extension Theorem for each $U \in \mathcal{B}(x)$ there exist $a_U \in A$ with $a_U(x) = 1$, $\|a_U\|_\infty \leq 1$, and $a_U(x') = 0$ for all $x' \in A \setminus U$. Since

$$\|(\xi \cdot a_U \mid \xi \cdot a_U)_A\| = \sup_{x' \in U} a_U^2(x')(\xi \mid \xi)_A(x')$$

and $(\xi \mid \xi)_A(x) = 0$, it follows that $\xi \cdot a_U \rightarrow 0$ in the norm induced by $(\cdot \mid \cdot)_A$. On the other hand, with respect to the supremum norm on Y we have $\|\xi \cdot a_U\| \geq 1$ as $(\xi \cdot a_U)(y) = 1$ for all $U \in \mathcal{B}(x)$. Hence, the norms $\|\cdot\| := \|(\cdot \mid \cdot)_A\|^{1/2}$ and $\|\cdot\|_\infty$ on $C(Y)$ are inequivalent. \square

Recall that the left action of $a \in A = C(\mathbb{A})$ on $\xi \in X_\Gamma = C(\mathbb{A} \times \Gamma)$ is given by $(a \cdot \xi)(x, \gamma) = a(p_1(x, \gamma))\xi(x, \gamma)$. Combining Proposition 3.3.2 and Lemma 4.4.7 we can now deduce the following.

Theorem 4.4.9. *Let (\mathbb{A}, Γ) be an injective iterated function system. If $\Delta C_\Gamma \neq \emptyset$, then there is no left A -valued inner product on X_Γ that is compatible with the left A -action and induces a norm equivalent to the supremum norm on X_Γ . In particular, X_Γ is not a bi-Hilbertian A - A -bimodule.*

Proof. If $\Delta C_\Gamma \neq \emptyset$, then Lemma 4.4.7 together with Proposition 4.4.4 implies that $p_1: \mathbb{A} \times \Gamma \rightarrow \mathbb{A}$ is not open. Since p_1 is used to define the left action, Proposition 3.3.2 implies that there is no A -valued left inner product that is compatible with the left action of A on X_Γ , and induces a norm equivalent to the supremum norm. Since the supremum norm on X_Γ is equivalent to the right-Hilbert module norm on X_Γ , the final statement follows from the definition of a bi-Hilbertian A - A -bimodule (Definition A.2.2). \square

Although Theorem 4.4.9 says nothing immediate about the C^* -algebras associated to X_Γ , it does say something about the interaction between C^* -correspondences and dynamics. The lack of a left inner product, suggests that the dynamics encoded by X_Γ is—in some sense—not continuously invertible. This is reminiscent of our use of the singular boundary in Section 4.1 to overcome the lack of continuity of the functions $\tau_w(a) := a \circ \gamma_w^{-1}$.

Remark 4.4.10. With a suitable adjustment to the definition of the critical boundary ΔC_Γ , a similar statement to Theorem 4.4.9 applies to the Kajiwara-Watatani correspondence E_Γ . Branch points must be taken into account, because the Kajiwara-Watatani correspondence corresponding to invertible systems do admit a left inner product inducing the

supremum norm (see Remark 2.5.10). We do not present this case since Theorem 4.4.9 already highlights the relationship between critical points and inner products.

Appendices

APPENDIX A

C*-correspondences and Cuntz-Pimsner algebras

A.1 | Hilbert C*-modules

We will assume some familiarity with Hilbert C*-modules and C*-correspondences. Accordingly, we only outline the basic definitions. The main purpose of this section is to establish notation. For an in-depth reference on Hilbert modules and C*-correspondences we suggest [Lan95], [RW98], and [MT05a].

Definition A.1.1. Let A be a C*-algebra. A *right pre-Hilbert A-module* (or an *inner product A-module*) is a vector space E over \mathbb{C} which is also a right A -module with right action $\cdot : E \times A \rightarrow A$, together with a map $(\cdot \mid \cdot)_A : E \times E \rightarrow A$ such that for all $\xi, \eta, \zeta \in E$, $a \in A$ and $\lambda, \mu \in \mathbb{C}$ we have

- (i) $(\xi \mid \lambda\eta + \mu\zeta)_A = \lambda(\xi \mid \eta)_A + \mu(\xi \mid \zeta)_A$;
- (ii) $(\xi \mid \eta \cdot a)_A = (\xi \mid \eta)_A a$;
- (iii) $(\xi \mid \eta)_A = (\eta \mid \xi)_A^*$;
- (iv) $(\xi \mid \xi)_A \geq 0$ as an element of A ; and
- (v) $(\xi \mid \xi)_A = 0$ implies $\xi = 0$.

The vector space E comes equipped with a norm given by $\|\xi\| := \|(\xi \mid \xi)_A\|^{1/2}$. If E is complete with respect to this norm we say that E is a *right Hilbert A-module*. The space $\overline{(E \mid E)_A}$ is a proper ideal of A . We say E is *full* if $\overline{(E \mid E)_A} = A$. We do not always assume that Hilbert C*-modules are full.

Left Hilbert A -modules can also be defined analogously to right Hilbert A -modules. The main differences being that we replace the right A -action on E by a left A -action; and the linearity of condition (i) and the A -linearity of condition (ii) both happen in the first variable. An inner product satisfying both (i) and (ii) in the left variable is instead denoted ${}_A(\cdot \mid \cdot)$. Usually we deal with right Hilbert A -modules, and often drop the adjective “right”. Sometimes we decorate a Hilbert module with its coefficient algebra. For example E_A denotes that E is a right Hilbert A -module and ${}_A E$ denotes that E is a left Hilbert A -module.

Unless otherwise specified we always assume that a C^* -algebra A comes equipped with its standard right Hilbert A -module structure. To be precise, A acts on itself by right multiplication and $(a \mid b)_A = a^*b$ for all $a, b \in A$. As above, we sometimes write A_A to indicate that we are considering A as a right Hilbert A -module.

If A_0 is a pre- C^* -algebra (satisfies all the axioms for a C^* -algebra except for completeness) and E_0 is a pre-Hilbert A_0 -module, then E_0 can be completed to obtain a Hilbert A -module E , where A is the completion A_0 (see [RW98, Lemma 2.16]). A Hilbert module E_A is *countably generated* if there exists a countable set $\{x_j\}_{j \in \mathbb{N}}$ such that the right A -linear span of $\{x_j\}$ is norm-dense in E_A .

Example A.1.2. Given a C^* -algebra A we write $\ell^2(A)$ for the standard Hilbert A -module

$$\ell^2(A) = \left\{ (a_i)_{i=1}^\infty \in \prod_{i=1}^\infty A \mid \sum_{i=1}^\infty a_i^* a_i \text{ converges in } A \right\}.$$

The right action of A on $\ell^2(A)$ is performed entry-wise, and the A -valued inner product is given by $((a_i) \mid (b_i))_A = \sum_{i=1}^\infty a_i^* b_i$. More generally, if $\{E_i\}_{i \in \mathbb{N}}$ is a collection of Hilbert A -modules, then we can form the direct sum

$$\bigoplus_{i=1}^\infty E_i = \left\{ (\xi_i)_{i=1}^\infty \in \prod_{i=1}^\infty E_i \mid \sum_{i=1}^\infty (\xi_i \mid \xi_i)_A \text{ converges in } A \right\}.$$

Again the right action of A is performed entry-wise, and the A -valued inner product is given by $((\xi_i) \mid (\eta_i))_A = \sum_{i=1}^\infty (\xi_i \mid \eta_i)_A$. △

Definition A.1.3. Let E and F be right Hilbert A -modules. A right A -linear operator $T: E \rightarrow F$ is said to be *adjointable* if there exists another (necessarily unique) linear operator $T^*: F \rightarrow E$ such that for all $\xi \in E$ and $\eta \in F$ we have $(T\xi \mid \eta)_A = (\xi \mid T^*\eta)_A$. We denote the collection of adjointable operators from E to F by $\text{End}_A(E, F)$. If $E = F$ then we write $\text{End}_A(E)$ for $\text{End}_A(E, E)$.

Adjointable operators are always bounded in operator norm and A -linear, but not all A -linear bounded operators on a Hilbert module are adjointable (see [RW98, Example 2.19]). The space $\text{End}_A(E)$ is itself C^* -algebra under composition (see [RW98, Proposition 2.21]).

Two right Hilbert A -modules E and F are *isomorphic* or *unitarily equivalent* if there exists an invertible adjointable operator $U \in \text{End}_A(E, F)$ such that $U^{-1} = U^*$. Such a U is necessarily isometric.

Definition A.1.4. Let E and F be right Hilbert A -modules. Given $\xi \in F$ and $\eta \in E$ we can form the *rank-one operator* $\Theta_{\xi, \eta}(\zeta) = \xi \cdot (\eta \mid \zeta)_A$ for $\zeta \in E$. Then $\Theta_{\xi, \eta}$ is adjointable with adjoint $\Theta_{\xi, \eta}^* = \Theta_{\eta, \xi}$. The closed linear span $\text{End}_A^0(E, F) := \overline{\text{span}}\{\Theta_{\xi, \eta} \mid \xi \in F, \eta \in E\}$ is a C^* -subalgebra of $\text{End}_A(E, F)$ called the *compact operators*. Again we write $\text{End}_A^0(E)$ in the case where $E = F$. Then $\text{End}_A^0(E)$ is a closed two-sided ideal of $\text{End}_A(E)$, as $\Theta_{\xi, \eta} T = \Theta_{\xi, T^* \eta}$ and $T \Theta_{\xi, \eta} = \Theta_{T\xi, \eta}$ for $T \in \text{End}_A(E)$.

Example A.1.5. When A is considered as a Hilbert A -module in the canonical way, $\text{End}_A(A)$ is the multiplier algebra $\mathcal{M}(A)$ of A and $\text{End}_A^0(A)$ is isomorphic to A . △

Suppose now that I is an ideal in a C^* -algebra A , and E is a right Hilbert A -module. We can form a new Hilbert A -module,

$$\begin{aligned} E \cdot I &:= \overline{\text{span}}\{\xi \cdot a \mid \xi \in E, a \in I\} \\ &= \{\xi \in E \mid (\xi \mid \xi)_A \in I\} \\ &= \{\xi \in E \mid (\xi \mid \eta)_A \in I, \text{ for all } \eta \in E\}. \end{aligned} \tag{A.1}$$

The equalities above follow from [Kat07, Proposition 1.3]. By [Kat07, Corollary 1.4] $E \cdot I$ can be viewed as a closed submodule of E which is invariant under the right A -action and left $\text{End}_A(E)$ action. By [FMR03, Lemma 2.6] we can identify $\text{End}_A^0(E \cdot I)$ with the C^* -subalgebra of $\text{End}_A^0(E)$ given by $\overline{\text{span}}\{\Theta_{\xi, \eta \cdot a} \mid \xi, \eta \in E_A, a \in I\}$.

Although Hilbert modules are analogous to Hilbert spaces, they do not typically admit orthonormal bases. One vestige from the setting of Hilbert spaces is the existence of frames. Frames in a Hilbert space are a generalisation of orthonormal bases, and find a variety of uses in signal analysis because of their robustness. Frames in the context of Hilbert modules were introduced by Frank and Larson [FL02], as well as Raeburn and Thompson [RT03] (in the non-unital case) at around the same time. Similar ideas to frames had appeared previously in the literature.

Definition A.1.6. Let A be a unital C^* -algebra. A *frame* for a countably generated right Hilbert A -module E_A is a sequence $(e_i)_{i \in \mathbb{N}}$ of elements of E_A such that

$$\sum_{i=1}^{\infty} e_i \cdot (e_i \mid \xi)_A = \xi \tag{A.2}$$

for all $\xi \in E_A$, where convergence is in norm.

The definition of a frame we have used is equivalent to the standard normalised tight frames of Frank and Larson by [FL02, Theorem 4.1] and [FL02, Example 3.1]. By definition a frame for E_A defines a countable generating set for E_A . It also follows from [FL02, Proposition 2.2] that $(e_i \mid e_i)_A \leq 1_A$, and so $\|e_i\| \leq 1$ for all $i \in \mathbb{N}$.

Each frame $(e_i)_{i \in \mathbb{N}}$ determines an A -linear isometry $V: E_A \rightarrow \ell^2(A)$ called the *analysis operator* by

$$V(\xi) = ((e_i \mid \xi)_A)_{i \in \mathbb{N}}.$$

The analysis operator is adjointable with adjoint $V^*: \ell^2(A) \rightarrow E_A$ —called the *synthesis operator*—given by

$$V^*((a_i)_{i \in \mathbb{N}}) = \sum_{i=1}^{\infty} e_i \cdot a_i.$$

Condition (A.2) is equivalent to the identity $V^*V = \text{id}_E$ (see [FL02, Theorem 4.1]).

The following is well known, and a version appears in [RT03].

Theorem A.1.7. *Let A be a unital C^* -algebra. Then every countably generated Hilbert A -module E_A admits a countable frame.*

Proof. Kasparov's Stabilisation Theorem [Kas80a, Theorem 2] implies that for any countably generated Hilbert A -module E_A , there is an isomorphism $E_A \oplus \ell^2(A) \cong \ell^2(A)$. Accordingly, there is an adjointable isometry $V: E_A \rightarrow \ell^2(A)$. Let $(x_i)_{i \in \mathbb{N}}$ denote the

standard orthonormal basis for $\ell^2(A)$ and let $e_i := V^*x_i$. Then,

$$\sum_{i=1}^k e_i \cdot (e_i \mid V^*\xi)_A = V^* \left(\sum_{i=1}^k x_i \cdot (x_i \mid V\xi)_A \right).$$

In $\ell^2(A)$ the sum $\sum_{i=1}^\infty x_i \cdot (x_i \mid V\xi)_A$ converges in norm to $V\xi$. Accordingly, $\sum_{i=1}^\infty e_i \cdot (e_i \mid V^*(\xi))_A$ converges in norm to $V^*V\xi = \xi$. \square

The frame condition (A.2) is equivalent to requiring that the sequence of operators $(\sum_{i=1}^n \Theta_{e_i, e_i})_{n \in \mathbb{N}}$ converges to id_A in the $*$ -strong topology on $\text{End}_A(E)$. We have the following characterisation of frames.

Proposition A.1.8. *Let E_A be a countably generated right Hilbert A -module. The following are equivalent:*

(i) $(e_i)_{i \in \mathbb{N}}$ is a frame for E_A ;

(ii) $(\sum_{i=1}^n \Theta_{e_i, e_i})_{n \in \mathbb{N}}$ is an increasing approximate identity for $\text{End}_A^0(E)$.

Proof. For (ii) \implies (i) note that for each $\xi \in E_A$ there exists $\zeta \in E_A$ such that $\xi = \Theta_{\zeta, \zeta}(\zeta)$ [RW98, Proposition 2.31]. Consequently,

$$\left\| \sum_{i=1}^n \Theta_{e_i, e_i} \xi - \xi \right\| = \left\| \sum_{i=1}^n \Theta_{e_i, e_i} \Theta_{\zeta, \zeta}(\zeta) - \Theta_{\zeta, \zeta}(\zeta) \right\| \rightarrow 0,$$

and hence $(\sum_{i=1}^n \Theta_{e_i, e_i})_{n \in \mathbb{N}}$ converges strictly to id_E .

For (i) \implies (ii) first observe that [Lan95, Lemma 4.1] implies that $\sum_{i=m}^n \Theta_{e_i, e_i} \geq 0$ for all $n \geq m$, so $(\sum_{i=1}^n \Theta_{e_i, e_i})_{n \in \mathbb{N}}$ is increasing. For $\xi, \eta \in E_A$,

$$\left(\sum_{i=1}^n \Theta_{e_i, e_i} \right) \Theta_{\xi, \eta} = \Theta_{\sum_{i=1}^n \Theta_{e_i, e_i}(\xi), \eta} \rightarrow \Theta_{\xi, \eta},$$

and an $\varepsilon/3$ -argument shows that $\sum_{i=1}^n \Theta_{e_i, e_i} T \rightarrow T$ for all $T \in \text{End}_A^0(E_A)$. \square

A.2 | C^* -correspondences

C^* -correspondences have found a variety of applications in the literature, from Rieffel's theory of induced representations and Morita equivalence, to Kasparov's KK -theory, and Pimsner's universal algebras for C^* -correspondences.

Definition A.2.1. Let A and B be C^* -algebras. A (right) A - B -correspondence is a pair (ϕ, E) consisting of right Hilbert B -module E together with a $*$ -homomorphism $\phi : A \rightarrow \text{End}_B(E)$. An A - A -correspondence is often called a C^* -correspondence over A .

The $*$ -homomorphism $\phi : A \rightarrow \text{End}_B(E)$ is usually referred to as the left action of A on E . If the left action is clear then we often just write $a \cdot \xi := \phi(a)\xi$ for $a \in A$ and $\xi \in E$. Similarly, we will sometimes simply write E for a C^* -correspondence instead of (ϕ, E) .

We say that an A - B -correspondence (ϕ, E) is *non-degenerate* if $\phi : A \rightarrow \text{End}_B(E)$ is a non-degenerate $*$ -homomorphism. If ϕ is non-degenerate it extends uniquely to a

$*$ -homomorphism $\tilde{\phi}: \text{End}_A(A) \rightarrow \text{End}_B(E)$. Similarly, we say that E is *faithful* if ϕ is injective.

We often take the view that C^* -correspondences are generalised morphisms of C^* -algebras. Indeed, if $\phi: A \rightarrow B$ is a $*$ -homomorphism, then (ϕ, B_B) is an A - B -correspondence under the identification of $\text{End}_B^0(B)$ with B .

C^* -correspondences are often called bimodules in the literature, but following [KPW04] we reserve this for the case where a correspondence is both a left and right C^* -correspondence.

Definition A.2.2. If E is both a right A - B correspondence and a left A - B correspondence then we call E an A - B -bimodule. An A - B -bimodule is said to be *bi-Hilbertian* if the norms induced by the left and right inner products are equivalent: that is if there exist constants $C, C' > 0$ such that for all $\xi \in E$,

$$C\|(\xi | \xi)_B\| \leq \|_A(\xi | \xi)\| \leq C'\|(\xi | \xi)_B\|.$$

A bi-Hilbertian A - B -bimodule E is called an *imprimitivity bimodule* or *invertible bimodule* if it is full in both the left and right inner products, and satisfies $_A(\xi | \eta)\zeta = \xi(\eta | \zeta)_B$ for all $\xi, \eta, \zeta \in E$. Two C^* -algebras A and B are said to be *Morita equivalent* if there exists an A - B -imprimitivity bimodule.

Let (ϕ, E) be an A - B -correspondence and (ψ, F) be a B - C -correspondence. We can form the *balanced tensor product* $E \otimes_B F$, which is defined to be the completion of the algebraic tensor product $E \odot F$ with respect to the norm induced by the C -valued inner product,

$$(\xi_1 \odot \eta_1 | \xi_2 \odot \eta_2)_C := (\eta_1 | \psi((\xi_1 | \xi_2)_B)\eta_2)_C,$$

defined for $\xi_1, \xi_2 \in E$ and $\eta_1, \eta_2 \in F$. Remarkably, $E \otimes_B F$ can also be realised as the quotient of $E \odot F$ by elements of the form $\xi \cdot b \odot \eta - \xi \odot \psi(b)\eta$ for all $\xi \in E, \eta \in F$ and $b \in B$ (see [Lan95, Proposition 4.5]). It follows that $E \otimes_B F$ becomes an A - C -correspondence with left action $\phi: A \rightarrow \text{End}_C(E \otimes_B F)$ satisfying $\tilde{\phi}(a)(\xi \otimes \eta) = (\phi(a)\xi) \otimes \eta$ and right action $(\xi \otimes \eta) \cdot c = \xi \otimes (\eta \cdot c)$. Often we denote the induced map $\tilde{\phi}$ by ϕ , unless there is potential for confusion.

For each $T \in \text{End}_B(E)$, there exists an operator $T \otimes \text{id} \in E \otimes_B F$ such that $(T \otimes \text{id})(\xi \otimes \eta) = T\xi \otimes \eta$ for all $\xi \in E_B$ and $\eta \in F_C$. Although it is well-known, it is a non-trivial fact that this formula extends to an adjointable operator on $E \otimes_B F$ (see [Lan95, p.42]). Moreover, the map $T \mapsto T \otimes \text{id}$ from $\text{End}_B(E)$ to $\text{End}_C(E \otimes_B F)$ is a unital $*$ -homomorphism, which is injective if and only if $\psi: B \rightarrow \text{End}_C(F)$ is injective.

It follows from [Lan95, Proposition 1.2] that for each $\xi \in E$ defines an operator $T_\xi \in \text{End}_C(F, E \otimes_B F)$ by $T_\xi(\eta) = \xi \otimes \eta$, whose adjoint satisfies $T_\xi^*(\eta \otimes \zeta) = \psi((\xi | \eta)_B)\zeta$. A straightforward computation shows that for all $\xi, \eta \in E$ we have,

$$T_\xi T_\eta^* = \Theta_{\xi, \eta} \otimes \text{id} \quad \text{and} \quad T_\xi^* T_\eta = (\xi | \eta)_B.$$

Typically, it is not true that $T \in \text{End}_B^0(E)$ implies $T \otimes \text{id} \in \text{End}_C^0(E \otimes_B F)$. Pimsner characterised precisely when this is the case.

Lemma A.2.3 ([Pim97, Corollary 3.7]). *Let (ϕ, E) be an A - B -correspondence and let (ψ, F) be a B - C -correspondence. For $T \in \text{End}_B^0(E)$ the following are equivalent:*

- (i) $T \otimes \text{id} \in \text{End}_C^0(E \otimes_B F)$; and

(ii) $T \in \text{End}_B^0(E \cdot I)$, where $I \triangleleft B$ is the ideal $I = \psi^{-1}(\text{End}_C^0(F))$.

Moreover, $T_{\xi_1} \Theta_{\eta_1, \eta_2} T_{\xi_2}^* = \Theta_{\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2}$ for all $\xi_1, \xi_2 \in E$, $\eta_1, \eta_2 \in F$ so $\text{End}_C^0(E \otimes_B F)$ is generated by elements of this form.

Although the map $T \mapsto T \otimes \text{id}$ from $\text{End}_B(E)$ to $\text{End}_C(E \otimes_B F)$ is only injective when ψ is injective, there is a subalgebra on which it is injective. Given an ideal I of a C^* -algebra A , recall that the *annihilator* of I is the ideal of A given by

$$I^\perp := \{a \in A \mid ab = 0 \text{ for all } b \in I\}.$$

Lemma A.2.4 (cf. [Kat04b, Lemma 4.7]). *Suppose (ϕ, E) is an A - B -correspondence and (ψ, F) is a B - C -correspondence. Let $J = \ker(\psi)^\perp$. Then the restriction of the map $T \mapsto T \otimes \text{id}: \text{End}_B(E) \rightarrow \text{End}_C(E \otimes_B F)$ to $\text{End}_B^0(E \cdot J)$ is injective. Moreover, $J \cap \overline{(E \mid E)}_B$ is the largest ideal in $\overline{(E \mid E)}_B$ with this property.*

Remark A.2.5. Although the first statement appears as [Kat04b, Lemma 4.7], the characterisation of J as the largest ideal in $\overline{(E \mid E)}_B$ satisfying the injectivity property did not appear. We note that the appearance of $\overline{(E \mid E)}_B$ is necessary if E is not full.

Proof of Lemma A.2.4. Let $T \in \text{End}_B^0(E \cdot J)$ and suppose that $T \otimes \text{id} = 0$. Then for all $\xi_1, \xi_2 \in E$ and $\eta_1, \eta_2 \in F$ we have,

$$0 = (\xi_1 \otimes \eta_1 \mid (T \otimes \text{id})(\xi_2 \otimes \eta_2))_C = (\eta_1 \mid \psi((\xi_1 \mid T\xi_2)_B)\eta_2)_C. \quad (\text{A.3})$$

Since this is true for all $\eta_1, \eta_2 \in F$ it follows that $\psi((\xi_1 \mid T\xi_2)_B) = 0$ for all $\xi_1, \xi_2 \in E$ so, $(\xi_1 \mid T\xi_2)_B \in \ker(\psi)$. Since $T \in \text{End}_B^0(E \cdot J)$ we have $(\xi_1 \mid T\xi_2)_B \in \ker(\psi)^\perp$. So $(\xi_1 \mid T\xi_2)_B = 0$ for all $\xi_1, \xi_2 \in E$. Hence, $T = 0$.

For the second statement suppose that I is an ideal in $\overline{(E \mid E)}_B$ with $I \cap \ker(\psi) \neq \{0\}$. Since $I \subseteq \overline{(E \mid E)}_B$ we can take $a \in I \setminus \{0\}$ such that $\psi(a) = 0$, and fix $\zeta, \eta \in E$ such that $\Theta_{\zeta, \eta \cdot a}$ is non-zero. Then for all $\xi_1, \xi_2 \in E$,

$$(\xi_1 \mid \Theta_{\zeta, \eta \cdot a} \xi_2)_B = (\xi_1 \mid \zeta)_B a^* (\eta \mid \xi_2)_B \in I \cap \ker(\psi).$$

Accordingly, (A.3) implies that $\Theta_{\zeta, \eta \cdot a} \otimes \text{id} = 0$. □

Just as with any reasonable algebraic object, it is possible to define morphisms of C^* -correspondences.

Definition A.2.6. Let (ϕ_E, E_A) be a correspondence over A and let (ϕ_F, F_B) be a correspondence over B . A *morphism* $(\pi, \psi): (\phi_E, E_A) \rightarrow (\phi_F, F_B)$ consists of a $*$ -homomorphism $\pi: A \rightarrow B$ and a linear map $\psi: E_A \rightarrow F_B$ such that for all $\xi, \eta \in E$ and $a \in A$ we have,

- (i) $\pi((\xi \mid \eta)_A) = (\psi(\xi) \mid \psi(\eta))_B$;
- (ii) $\phi_F(\pi(a))\psi(\xi) = \psi(\phi_E(a)\xi)$; and
- (iii) $\psi(\xi)\pi(a) = \psi(\xi \cdot a)$.

A morphism (π, ψ) is said to be *injective* if π is injective. In the case where $(\phi_F, F) = (\text{id}, B_B)$ a morphism $(\pi, \psi): (\phi_E, E_A) \rightarrow (\text{id}, B_B)$ is called a *representation* of (ϕ_E, E) in B , and we instead write $(\pi, \psi): (\phi_E, E_A) \rightarrow B$.

If $(\pi, \psi): (\phi_E, E_A) \rightarrow (\phi_F, F_B)$ is a morphism of C^* -correspondences, then

$$\|\psi(\xi)\|^2 = \|(\psi(\xi) \mid \psi(\xi))_B\| \leq \|\pi((\xi \mid \xi)_A)\| = \|\xi\|^2$$

for all $\xi \in E_A$. If π is injective we have equality throughout, so ψ is isometric.

Lemma A.2.7 ([KPW98, Lemma 2.2]). *If $(\pi, \psi): (\phi_E, E_A) \rightarrow (\phi_F, F_B)$ is a morphism of C^* -correspondences then there is an induced $*$ -homomorphism $\psi^{(1)}: \text{End}_A^0(E_A) \rightarrow \text{End}_B^0(F_B)$ satisfying*

$$\psi^{(1)}(\Theta_{\xi, \eta}) = \Theta_{\psi(\xi), \psi(\eta)},$$

for all $\xi, \eta \in E_A$. Moreover, if (π, ψ) is injective, then so is $\psi^{(1)}$.

Proof. The existence of $\psi^{(1)}$ follows from the arguments of [KPW98, Lemma 2.2]. If (π, ψ) is injective, then for all $\xi, \eta \in E_A$ we have

$$\begin{aligned} \|\Theta_{\psi(\xi), \psi(\eta)}\psi(\zeta)\|^2 &= \|(\psi(\zeta) \mid \psi(\eta))_B(\psi(\xi) \mid \psi(\xi))_B(\psi(\eta) \mid \psi(\zeta))_B\| \\ &= \|\pi((\zeta \mid \eta)_A(\xi \mid \xi)_A(\eta \mid \zeta)_A)\| = \|\Theta_{\xi, \eta}\zeta\|^2, \end{aligned}$$

for all $\zeta \in E_A$. Hence, $\|\Theta_{\psi(\xi), \psi(\eta)}\| = \|\Theta_{\xi, \eta}\|$. It follows that $\psi^{(1)}$ is injective. \square

If (ϕ, E) is a correspondence over A we denote by $E^{\otimes n}$ the n -fold balanced tensor product,

$$E^{\otimes n} := \underbrace{E \otimes_A E \otimes_A \cdots \otimes_A E}_{n\text{-times}}.$$

It is routine to check that a morphism $(\pi, \psi): (\phi_E, E_A) \rightarrow (\phi_F, F_B)$ induces a morphism $(\pi, \psi^n): (\phi_E, E_A^{\otimes n}) \rightarrow (\phi_F, F_B^{\otimes n})$ satisfying $\psi^n(\xi_1 \otimes \cdots \otimes \xi_n) = \psi(\xi_1) \otimes \cdots \otimes \psi(\xi_n)$ for all $\xi_1, \dots, \xi_n \in E_A$. It follows from Lemma A.2.7 that there is an induced $*$ -homomorphism $\psi^{(n)}: \text{End}_A^0(E_A^{\otimes n}) \rightarrow \text{End}_B^0(F_B^{\otimes n})$. When $(\pi, \psi): (\phi_E, E_A) \rightarrow B$ is a representation we see that $\psi^n(\xi_1 \otimes \cdots \otimes \xi_n) = \psi(\xi_1) \cdots \psi(\xi_n)$ and $\psi^{(n)}(\Theta_{\xi, \eta}) = \psi^n(\xi)\psi^n(\eta)^*$.

A.3 | Toeplitz and Cuntz-Pimsner algebras

In [Pim97], Pimsner associated to a C^* -correspondence E two C^* -algebras \mathcal{T}_E and \mathcal{O}_E , now known the *Toeplitz algebra (or Toeplitz-Pimsner)* and *Cuntz-Pimsner algebra* of E , respectively. The Cuntz-Pimsner algebra should be thought of as a “crossed-product by the correspondence E ”. The construction of \mathcal{O}_E simultaneously generalises Cuntz-Krieger algebras [CK80], graph C^* -algebras [Kum+97], and crossed products by a single automorphism, as well as many other classes of C^* -algebras which had been previously considered in the literature.

Muhly and Solel [MS98] extended Pimsner’s work to include relative Cuntz-Pimsner algebras $\mathcal{O}_{E, J}$, where J is an ideal in A which acts compactly on E . They also considered the non-self-adjoint tensor algebras associated to a C^* -correspondence together with their C^* -envelopes.

Pimsner’s algebras were later carefully studied and generalised by Katsura in [Kat04b]. In Pimsner’s original approach, he only considered full C^* -correspondences with non-degenerate, injective left actions. In practice however, one is often faced with correspondences which do not satisfy these hypotheses. This is realised in the correspondences

associated to topological quivers considered in Chapter 3 and Chapter 4. Katsura removed Pimsner's hypotheses by modifying the so called covariance ideal, which we describe below.

Let $(\pi, \psi): (\phi, E_A) \rightarrow B$ be a representation of a C^* -correspondence (π, E_1) over A in a C^* -algebra B . We denote by $C^*(\pi, \psi)$ the C^* -subalgebra of B generated by $\pi(A) \cup \psi(E)$.

Definition A.3.1. The *Toeplitz algebra* \mathcal{T}_E of a C^* -correspondence (ϕ, E_A) over A is the (unique up to isomorphism) C^* -algebra generated by a representation $(j_A, j_E): (\phi, E_A) \rightarrow \mathcal{T}_E$ satisfying the following universal property: if $(\pi, \psi): (\phi, E_A) \rightarrow B$ is a representation, then there exists a unique $*$ -homomorphism $\pi \times \psi: \mathcal{T}_E \rightarrow C^*(\pi, \psi)$ such that the diagram

$$\begin{array}{ccc} (\phi, E_A) & \xrightarrow{(j_A, j_E)} & \mathcal{T}_E \\ & \searrow (\pi, \psi) & \downarrow \pi \times \psi \\ & & C^*(\pi, \psi) \end{array}$$

commutes. Note that $\pi \times \psi$ can be thought of as a representation of $(\text{id}, \mathcal{T}_E)$ in $C^*(\pi, \psi)$.

The algebra \mathcal{T}_E admits a concrete description via a Fock-module representation. Given a C^* -correspondence (ϕ, E_A) over A , define $E^{\otimes 0} := A$, the *Fock module* of E to be the Hilbert A -module

$$\mathcal{F}(E) := \bigoplus_{n=0}^{\infty} E^{\otimes n}.$$

Then $\mathcal{F}(E)$ admits a left action $\phi_{\infty}: A \rightarrow \text{End}_A(\mathcal{F}(E))$ given by

$$\phi_{\infty}(a)(\xi) = \begin{cases} \phi(a)\xi & \text{if } \xi \in E^{\otimes n}, n > 0, \\ a\xi & \text{if } \xi \in A. \end{cases}$$

For each $\xi \in E$ we define an operator T_{ξ} on $\mathcal{F}(E)$ by

$$T_{\xi}(\eta) = \begin{cases} \xi \otimes \eta & \text{if } \eta \in E^{\otimes n}, n > 1, \\ \xi \cdot \eta & \text{if } \eta \in A = E^{\otimes 0}. \end{cases} \quad (\text{A.4})$$

The map T_{ξ} extends to an adjointable operator on $\mathcal{F}(E)$ which we call a *creation operator*. The adjoint T_{ξ}^* is known as an *annihilation operator* and satisfies

$$T_{\xi}^*(\eta) = \begin{cases} \phi((\xi \mid \eta_1)_A)\eta_2 & \text{if } \eta = \eta_1 \otimes \eta_2 \in E \otimes E^{\otimes n}, \\ 0 & \text{if } \eta \in E^0. \end{cases}$$

It follows from [Kat04b, Proposition 4.3], that the pair $(\phi_{\infty}, \xi \mapsto T_{\xi})$ defines an injective representation of (ϕ, E_A) in $\text{End}_A(\mathcal{F}(E))$. Moreover, [Kat04b, Proposition 6.5] implies that $C^*(\phi_{\infty}, \xi \mapsto T_{\xi})$ is isomorphic to \mathcal{T}_E .

Let (ϕ, E_A) be a C^* -correspondence over A , and suppose that $J \triangleleft \phi^{-1}(\text{End}_A^0(E))$. If $(\pi, \psi): (\phi, E_A) \rightarrow B$ is a representation, then we can map $a \in J$ into B via both π and $\psi^{(1)} \circ \phi$. Representations for which these maps agree on J are distinguished.

Definition A.3.2. Let (ϕ, E_A) be a C^* -correspondence over A , and suppose that $J \triangleleft \phi^{-1}(\text{End}_A^0(E))$. A representation $(\pi, \psi): (\phi, E_A) \rightarrow B$ is said to be J -covariant if

$$\pi(a) = \psi^{(1)}(\phi(a)) \quad (\text{A.5})$$

for all $a \in J$.

Remark A.3.3. Let (ϕ, E_A) be a countably generated C^* -correspondence over A , and let $(e_i)_{i \in \mathbb{N}}$ be a frame for E_A . Since $(\Theta_{e_i, e_i})_{i \in \mathbb{N}}$ acts as an approximate identity for $\text{End}_A^0(E)$, for all $a \in J$ we have $\phi(a) = \sum_{i=1}^{\infty} \Theta_{\phi(a)e_i, e_i}$. As observed in [Pim97, Remark 3.9], a representation $(\pi, \psi): (\phi, E_A) \rightarrow B$ is J -covariant if and only if

$$\sum_{i=1}^{\infty} \psi(\phi(a)e_i)\psi(e_i)^* = \pi(a) \quad (\text{A.6})$$

for all $a \in J$.

Definition A.3.4 (cf. [FMR03, Proposition 1.3]). The J -relative Cuntz-Pimsner algebra $\mathcal{O}_{E,J}$ of a C^* -correspondence (ϕ, E_A) over A is the C^* -algebra generated by a representation $(i_A^J, i_E^J): (\phi, E_A) \rightarrow \mathcal{O}_{E,J}$ that satisfies the following universal property: if $(\pi, \psi): (\phi, E_A) \rightarrow B$ is a J -covariant representation, then there exists a unique $*$ -homomorphism $\pi \times \psi: \mathcal{O}_{E,J} \rightarrow C^*(\pi, \psi)$ that makes the following diagram commute:

$$\begin{array}{ccc} (\phi, E_A) & \xrightarrow{(i_A^J, i_E^J)} & \mathcal{O}_{E,J} \\ & \searrow (\pi, \psi) & \downarrow \pi \times \psi \\ & & C^*(\pi, \psi) \end{array}$$

It follows from the proof of [FMR03, Proposition 1.3] that $\mathcal{O}_{E,J}$ is the quotient of \mathcal{T}_E by the ideal generated by $\{j_A(a) - j_E^{(1)} \circ \phi(a) \mid a \in J\}$. Relative Cuntz-Pimsner algebras were first introduced by Muhly and Solel [MS98, Definition 2.18], who instead described $\mathcal{O}_{E,J}$ in terms of the Fock-module representation of \mathcal{T}_E . If $J = \{0\}$ then $\mathcal{O}_{E,\{0\}}$ is isomorphic to \mathcal{T}_E .

It follows from [MS98, Proposition 2.21] that if $J \cap \ker(\phi) = \{0\}$ then the representation $(i_A^J, i_E^J): (\phi, E_A) \rightarrow \mathcal{O}_{E,J}$ is injective. On the other hand, if $(\pi, \psi): (\phi, E_A) \rightarrow B$ is a J -covariant representation, then $J \cap \ker(\phi) \subseteq \ker(\pi)$. Consequently, an injective J -covariant representation of (ϕ, E_A) exists if and only if $J \subseteq \ker(\phi)^\perp$ (cf. [KL13, Proposition 4.12]). Following Katsura [Kat04b], we introduce the following ideal of A .

Definition A.3.5. The *Katsura ideal* or *covariance ideal* is the ideal of A given by

$$\begin{aligned} I_E &:= \phi^{-1}(\text{End}_A^0(E)) \cap \ker(\phi)^\perp \\ &= \{a \in A \mid \phi(a) \in \text{End}_A^0(E) \text{ and } ab = 0 \text{ for all } b \in \ker(\phi)\}. \end{aligned} \quad (\text{A.7})$$

Note that Lemma A.2.3 and Lemma A.2.4 imply that the map $T \mapsto T \otimes \text{id}$ from $\text{End}_A(E)$ to $\text{End}_A(E \otimes_A E)$ is injective and takes values in $\text{End}_A^0(E)$ when it is restricted to $\text{End}_A^0(E \cdot I_E)$. If $J \triangleleft I_E$, then by the above discussion, the universal representation $(i_A^J, i_E^J): (\phi, E_A) \rightarrow \mathcal{O}_{E,J}$ is injective.

Definition A.3.6. Let (ϕ, E_A) be a C^* -correspondence over A and let I_E be the associated Katsura ideal. We call \mathcal{O}_{E, I_E} the *Cuntz-Pimsner algebra* of (ϕ, E_A) and denote it by \mathcal{O}_E . We denote the universal representation by $(i_A, i_E): (\phi, E_A) \rightarrow \mathcal{O}_E$.

If the left action $\phi: A \rightarrow \text{End}_A(E)$ is injective, then $I_E = \phi^{-1}(\text{End}_A^0(E))$. In this case Katsura's Cuntz-Pimsner algebra agrees with Pimsner's construction [Pim97] when E is full.

In Chapter 3, we make use of the following notion (cf. [Kat04b, Definition 5.8]).

Lemma A.3.7. *Suppose that $(\pi, \psi): (\phi, E_A) \rightarrow B$ is an injective representation of a C^* -correspondence over A . Then $I_{(\pi, \psi)} := \{a \in A \mid \pi(a) \in \psi^{(1)}(\text{End}_A^0(E))\}$ is an ideal of A contained in I_E , and $\pi(a) = \psi^{(1)}(\phi(a))$ for all $a \in I_{(\pi, \psi)}$.*

Proof. It follows [Kat04b, Proposition 3.3] that $I_{(\pi, \psi)}$ is contained in I_E , and that for $a \in I_{(\pi, \psi)}$ we have $\pi(a) = \psi^{(1)}(\phi(a))$. Using the construction of $\psi^{(1)}$ at the second equality, for $b \in A$ we calculate

$$\pi(ba) = \pi(b)\psi^{(1)}(\phi(a)) = \psi^{(1)}(\phi(b)\phi(a)) = \psi^{(1)}(\phi(ba)) \in I_{(\pi, \psi)}. \quad \square$$

Definition A.3.8. We call the ideal $I_{(\pi, \psi)}$ of Lemma A.3.7 the *ideal of covariance* for the representation (π, ψ) .

Lemma A.3.9 ([Kat07, Corollary 11.4]). *Let (ϕ, E_A) be a C^* -correspondence over A , and suppose that J is an ideal contained in I_E . Let $(i_A^J, i_E^J): (\phi, E_A) \rightarrow \mathcal{O}_{E, J}$ denote the universal J -covariant representation. Then $I_{(i_A^J, i_E^J)} = J$.*

Although we do not use it, it is worth noting that Katsura showed that every relative Cuntz-Pimsner algebra $\mathcal{O}_{E, J}$ can be realised as the Cuntz-Pimsner algebra of a modified C^* -correspondence E_{ω_J} (see [Kat07, Proposition 11.3]). In particular, the Cuntz-Pimsner and Toeplitz algebras of Pimsner, together with their augmented versions, can be realised in Katsura's framework.

Definition A.3.10. A representation $(\pi, \psi): (\phi, E_A) \rightarrow B$ of an A - A -correspondence (ϕ, E_A) is said to *admit a gauge action* if there is a strongly continuous \mathbb{T} -action $\gamma: \mathbb{T} \rightarrow \text{Aut}(C^*(\pi, \psi))$ satisfying $\gamma_z(\pi(a)) = \pi(a)$ and $\gamma_z(\psi(\xi)) = z\psi(\xi)$ for all $a \in A$, $\xi \in E_A$ and $z \in \mathbb{T}$. If (π, ψ) admits a gauge action $\gamma: \mathbb{T} \rightarrow \text{Aut}(C^*(\pi, \psi))$, then we call the fixed point algebra $C^*(\pi, \psi)^{\mathbb{T}}$ the *core* of $C^*(\pi, \psi)$.

Universality implies that the universal representations (j_A, j_E) of \mathcal{T}_E and (i_A, i_E) of \mathcal{O}_E both admit gauge actions. A particularly useful feature of the gauge action is the following theorem.

Theorem A.3.11 ([Kat04b, Theorem 6.4] Gauge-invariant Uniqueness Theorem). *Suppose that (π, ψ) is an I_E -covariant representation of a C^* -correspondence (ϕ, E_A) over A . Then the induced $*$ -homomorphism $\pi \times \psi: \mathcal{O}_E \rightarrow C^*(\pi, \psi)$ is an isomorphism if and only if (π, ψ) is injective and admits a gauge action.*

Using the Gauge-Invariant Uniqueness Theorem, we obtain the following functoriality results for the Toeplitz algebra and Cuntz-Pimsner algebra of a correspondence.

Lemma A.3.12. *Suppose that $(\pi, \psi): (\phi_E, E_A) \rightarrow (\phi_F, F_B)$ is a morphism of correspondences. Let $(j_A, j_E): (\phi_E, E_A) \rightarrow \mathcal{T}_E$ and $(j_B, j_F): (\phi_F, F_B) \rightarrow \mathcal{T}_F$ be the associated universal representations. Then there is a unique $*$ -homomorphism $\pi \times \psi: \mathcal{T}_E \rightarrow \mathcal{T}_F$ such that $(\pi \times \psi) \circ j_A = j_B$ and $(\pi \times \psi) \circ j_E = j_F$. Moreover, if (π, ψ) is injective then so is $(\pi \times \psi)$.*

Proof. Since $(j_B \circ \pi, j_F \circ \psi)$ defines a representation of (ϕ_E, E_A) in \mathcal{T}_F , the universal property of \mathcal{T}_E gives a $*$ -homomorphism $\pi \times \psi: \mathcal{T}_E \rightarrow \mathcal{T}_F$ with the desired properties.

If $j_B \circ \pi(a) \in (j_F \circ \psi)^{(1)}(\text{End}_A^0(E))$, then $\pi(a) \in I_{(j_B, j_F)} = \{0\}$ by [Kat04b, Proposition 4.10]. Injectivity of π implies that $I_{(j_B \circ \pi, j_F \circ \psi)} = 0$. An application of the Gauge-Invariant Uniqueness Theorem [Kat04b, Theorem 6.2] for \mathcal{T}_E now implies that $\pi \times \psi$ is injective. \square

For a Cuntz-Pimsner version of Lemma A.3.12 we require the following notion.

Definition A.3.13. A morphism $(\pi, \psi): (\phi_E, E_A) \rightarrow (\phi_F, F_B)$ of correspondences is said to be J -covariant if $\psi^{(1)} \circ \phi_E(a) = \phi_F \circ \pi(a)$ for all $a \in J$, where J is an ideal contained in $\phi_E^{-1}(\text{End}_A^0(E))$. An I_E -covariant ideal is simply said to be *covariant*.

If $(\pi, \psi): (\phi_E, E_A) \rightarrow (\phi_F, F_B)$ is J -covariant then $\pi(a) \in \phi_F^{-1}(\text{End}_B^0(F))$.

Lemma A.3.14 ([Bre10b, Corollary 1.5]). *Suppose that $(\pi, \psi): (\phi_E, E_A) \rightarrow (\phi_F, F_B)$ is a covariant morphism of correspondences, satisfying $\pi(I_E) \subseteq I_F$. Let $(i_A, i_E): (\phi_E, E_A) \rightarrow \mathcal{O}_E$ and $(i_B, i_F): (\phi_F, F_B) \rightarrow \mathcal{O}_F$ be the associated universal covariant representations. Then there is a unique $*$ -homomorphism $\pi \times \psi: \mathcal{O}_E \rightarrow \mathcal{O}_F$ such that $(\pi \times \psi) \circ i_A = i_B$ and $(\pi \times \psi) \circ i_E = i_F$. Moreover, if (π, ψ) is injective then so is $(\pi \times \psi)$.*

Remark A.3.15. We note that if ϕ_F is injective then the hypothesis $\pi(I_E) \subseteq I_F$ is automatically satisfied.

Proof of Lemma A.3.14. Since $\pi(I_E) \subseteq I_F$, for each $a \in I_E$ we have,

$$i_F^{(1)} \circ \psi^{(1)} \circ \phi_E(a) = i_F^{(1)} \circ \phi_F \circ \pi(a) = i_B \circ \pi(a).$$

Consequently, $(j_B \circ \pi, j_E \circ \psi)$ defines an I_E -covariant representation of (ϕ_E, E_A) in \mathcal{O}_F . The universal property of \mathcal{O}_E gives the desired $*$ -homomorphism $\pi \times \psi: \mathcal{O}_E \rightarrow \mathcal{O}_F$. The final statement follows from the Gauge Invariant Uniqueness Theorem (Theorem A.3.11) since $\pi \times \psi: \mathcal{O}_E \rightarrow \mathcal{O}_F$ preserves the gauge action. \square

The following generalisation of Remark A.3.3 proves useful when trying to determine whether a morphism is J -covariant. The author has not been able to find this result in the literature.

Lemma A.3.16. *Suppose that $(\pi, \psi): (\phi_E, E_A) \rightarrow (\phi_F, F_B)$ is a morphism of correspondences over a unital C^* -algebra A , and suppose that $J \triangleleft \phi_E^{-1}(\text{End}_A^0(E))$ satisfies $\pi(J) \subseteq \phi_F^{-1}(\text{End}_B^0(F))$. Assume that E_A is countably generated as a right A -module. Then (π, ψ) is J -covariant if and only if for some (and hence every) frame $(e_i)_{i=1}^\infty$ of E_A , the sequence $(\sum_{i=1}^n \Theta_{\psi(e_i), \psi(e_i)})_{n \in \mathbb{N}}$ is an approximate identity for $\phi_F(\pi(J))$.*

Proof. Fix $a \in J$ and a frame $(e_i)_{i \in \mathbb{N}}$ of E_A . Since $\phi_E(a) \in \text{End}_A^0(E)$ we have $\phi_E(a) = \sum_{i=1}^{\infty} \Theta_{\phi_E(a)e_i, e_i}$ with convergence in norm. Then,

$$\psi^{(1)} \circ \phi_E(a) = \psi^{(1)} \left(\sum_{i=1}^{\infty} \Theta_{\phi_E(a)e_i, e_i} \right) = \sum_{i=1}^{\infty} \Theta_{\psi(\phi_E(a)e_i), \psi(e_i)} = \sum_{i=1}^{\infty} \Theta_{\phi_F(\pi(a))\psi(e_i), \psi(e_i)}.$$

Thus, (π, ψ) is J -covariant if and only if $\sum_{i=1}^{\infty} \Theta_{\phi_F(\pi(a))\psi(e_i), \psi(e_i)} = \phi_F(\pi(a))$ for all $a \in J$. Since $\sum_{i=1}^n \Theta_{\phi_F(\pi(a))\psi(e_i), \psi(e_i)} = \phi_F(\pi(a)) \sum_{i=1}^n \Theta_{\psi(e_i), \psi(e_i)}$ we see that (π, ψ) is J -covariance if and only if

$$\left\| \phi_F(\pi(a)) \left(\sum_{i=1}^n \Theta_{\psi(e_i), \psi(e_i)} \right) - \phi_F(\pi(a)) \right\| \rightarrow 0$$

as $n \rightarrow \infty$, for all $a \in J$. □

We finish our discussion of Cuntz-Pimsner algebras with the following K -theoretic result. For the fundamentals of K -theory and KK -theory see [WO93], [RLL00], [Bla98], and [Kas80b].

Theorem A.3.17 ([Kat04b, Theorem 8.8], [Pim97, Theorem 4.9]). *Let (ϕ, E_A) be a C^* -correspondence over A . Let $\iota_{I, A*}$ is the map on K -theory induced by the inclusion of I into A , and let π_* be the map on K -theory induced by the inclusion of A into \mathcal{O}_E . Denote the class of the even Kasparov module $(\phi, E_A, 0)$ in $KK(I_E, A)$ by $[E]$, and let \otimes denote the Kasparov product. Then the six-term sequence*

$$\begin{array}{ccccc} K_0(I_E) & \xrightarrow{\otimes(\iota_{I, A*} - [E])} & K_0(A) & \xrightarrow{\pi_*} & K_0(\mathcal{O}_E) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathcal{O}_E) & \xleftarrow{\pi_*} & K_1(A) & \xleftarrow{\otimes(\iota_{I, A*} - [E])} & K_1(I_E) \end{array}$$

is exact.

Despite the somewhat daunting appearance of the Kasparov product in Theorem A.3.17, in practice the product $\cdot \otimes [E]$ is often quite computable. We note that in general $K_*(A) \cong K_*(\mathcal{T}_E)$ [Kat04b, Proposition 8.2], and if in addition A is separable and E is countably generated then A is KK -equivalent to \mathcal{T}_E [Pim97, Theorem 4.4].

APPENDIX B

Groupoids and their C^* -algebras

Groupoids and their C^* -algebras are an essential tool in the arsenal of many C^* -algebraists and have found a wide variety of applications in the literature, especially within the context of topological dynamics. The primary purpose of this appendix is to both establish notation and terminology, and also to highlight some key groupoid concepts that we use. Most of these results are standard in the literature. For a more in-depth discussion of groupoids and their C^* -algebras see [Wil19], [Ren80], and [Pat99].

B.1 | Groupoids

We begin with the definition of a groupoid.

Definition B.1.1. A *groupoid* is a set G together with a partially defined multiplication from a subset $G^{(2)} \subseteq G \times G$ —called the *composable pairs*—to G , $(g_1, g_2) \mapsto g_1 g_2$, and a map from G to G given by $g \mapsto g^{-1}$ which satisfy:

- (i) (Associativity) for all $(g_1, g_2) \in G^{(2)}$ and $(g_2, g_3) \in G^{(2)}$ we have $(g_1, g_2 g_3)$ and $(g_1 g_2, g_3)$ in $G^{(2)}$, and $(g_1 g_2) g_3 = g_1 (g_2 g_3)$;
- (ii) (Involution) for all $g \in G$ we have $(g^{-1})^{-1} = g$; and
- (iii) (Identity) for all $g \in G$ we have $(g, g^{-1}) \in G^{(2)}$, and for all $(g_1, g_2) \in G^{(2)}$ we have $g_1 g_2 g_2^{-1} = g_1$ and $g_1^{-1} g_1 g_2 = g_2$.

Although each $g \in G$ has an inverse, $g g^{-1}$ and $g^{-1} g$ may not be equal. A groupoid potentially has many elements which act like identities. We call the collection

$$G^{(0)} := \{g g^{-1} \mid g \in G\} = \{g^{-1} g \mid g \in G\}.$$

the *unit space* of G and refer to elements of $G^{(0)}$ as *units*.

Example B.1.2 (Groups). Every group G is a groupoid with $G^{(0)}$ consisting of the unique identity element. ^a ▲

We can now define maps $r : G \rightarrow G^{(0)}$ and $s : G \rightarrow G^{(0)}$ by $r(g) = g g^{-1}$ and $s(g) = g^{-1} g$. We call r the *range map* and s the *source map*. We also refer to $r(g)$ and

^aIt is sometimes said that groupoids are “groups with an identity crisis” [source unknown].

$s(g)$ as the *range* and *source* of $g \in G$ respectively. For each $A \subseteq G^{(0)}$ we adopt the notation

$$G^A := \{g \in G \mid r(g) \in A\}$$

$$G_A := \{g \in G \mid s(g) \in A\}$$

$$G|_A := G_A^A := G_A \cap G^A.$$

Note that $G|_A$ is itself a groupoid with unit space A . We call $G|_A$ the *restriction of G to A* . If $A = \{u\}$ is a singleton then we write $G^u := G^{\{u\}}$ and $G_u := G_{\{u\}}$.

We are exclusively interested in groupoids equipped with a topology.

Definition B.1.3. A *topological groupoid* is a groupoid G equipped with a topology which makes inversion and multiplication continuous (we equip $G^{(2)}$ with the subspace topology from $G \times G$).

In a topological groupoid both $r : G \rightarrow G^{(0)}$ and $s : G \rightarrow G^{(0)}$ are continuous. If G is Hausdorff, then $G^{(0)}$ is closed in G [Wil19, Lemma 1.6]. We deal almost exclusively with second-countable locally compact Hausdorff groupoids. It is not always the case that $G^{(0)}$ is open in G ; so we give such groupoids a name.

Definition B.1.4. A locally compact Hausdorff groupoid G is said to be *r -discrete* if $G^{(0)}$ is open in G . We say that G is *étale* if G is r -discrete and $r : G \rightarrow G^{(0)}$ and $s : G \rightarrow G^{(0)}$ are both open maps.

It is straightforward to check that if G is r -discrete then both G^u and G_u are discrete for all $u \in G^{(0)}$. However, the converse is not necessarily true (see Example 3.2.8). If G is a locally compact Hausdorff groupoid and A is an open or closed subset of G then $G|_A$ is a locally compact Hausdorff groupoid.

Like all good algebraic objects, groupoids have an appropriate notion of homomorphism.

Definition B.1.5. A groupoid homomorphism from G to H is a map $\phi : G \rightarrow H$ such that $(\phi(g_1), \phi(g_2)) \in H^{(2)}$ for all $(g_1, g_2) \in G^{(2)}$ and $\phi(g_1 g_2) = \phi(g_1) \phi(g_2)$. We say ϕ is an isomorphism if it is bijective (in which case ϕ^{-1} is a homomorphism). If G and H are topological groupoids we insist that ϕ be continuous, and that isomorphisms are also homeomorphisms.

Example B.1.6 (Equivalence relations). Suppose that $R \subseteq X \times X$ is an equivalence relation on a set X . We define a set of composable pairs $R^{(2)} \subseteq R \times R$ to consist pairs $((x, y), (y, z))$. Define a multiplication on composable pairs by $(x, y)(y, z) = (x, z)$, and an inversion on R by $(x, y)^{-1} = (y, x)$. Then R is a groupoid with unit space given by the diagonal $\{(x, x) \mid x \in X\}$, which we often identify with X . The range map is given by $r(x, y) = (x, x)$ and the source map is given by $s(x, y) = (y, y)$. If X is a locally compact Hausdorff space then R becomes a locally compact Hausdorff groupoid when equipped with subspace topology inherited from $X \times X$.

As groupoids, equivalence relations are often equipped with a topology that is distinct from the subspace topology inherited from $X \times X$. This is usually done to give R more desirable topological properties such as étaleness. \triangleleft

B.2 | Groupoid C^* -algebras

To associate a C^* -algebra to a groupoid G we first construct a convolution product on $C_c(G)$. With locally compact Hausdorff groups, the convolution is usually performed by integrating against a Haar measure. For groupoids on the other hand, we have a measure for each element of the unit space.

Definition B.2.1. Let G be a locally compact Hausdorff groupoid. A *Haar system* on G consists of a family of positive Radon measures $\mu = \{\mu^u\}_{u \in G^{(0)}}$ on G such that:

(i) for each $u \in G^{(0)}$ we have $\text{supp } \mu^u = G^u$;

(ii) for all $f \in C_c(G)$ the map,

$$u \mapsto \int_G f d\mu^u,$$

is continuous; and

(iii) for all $f \in C_c(G)$ and $g_1 \in G$ we have,

$$\int_G f(g_1 g_2) d\mu^{s(g_1)}(g_2) = \int_G f(g_3) d\mu^{r(g_1)}(g_3).$$

A locally compact Hausdorff groupoid equipped with a Haar system is sometimes called a *Haar groupoid*.

Unlike groups, locally compact Hausdorff groupoids may not admit Haar systems (see [Dei18, Proposition 3.2]). However, étale groupoids admit a Haar system given by counting measures on the range fibres G^u .

Proposition B.2.2 ([Ren80, Proposition I.2.8] [Wil19, Proposition 1.29]). *If G is a locally compact Hausdorff groupoid, then the following are equivalent:*

(i) G is étale;

(ii) G is r -discrete and admits a Haar system consisting of counting measures; and

(iii) r (or equivalently s) is a local homeomorphism: for each $g \in G$ there is an open neighbourhood $U \subseteq G$ of g such that $r(U)$ is open in $G^{(0)}$ and $r|_U : U \rightarrow r(U)$ is a homeomorphism.

Given a Haar groupoid G we can define a convolution product on the compactly supported continuous functions $C_c(G)$.

Proposition B.2.3 ([Wil19, Proposition 1.34]). *Let G be a locally compact Hausdorff groupoid with Haar system $\mu = \{\mu^u\}_{u \in G^{(0)}}$. Then the vector space $C_c(G)$ is a complex $*$ -algebra with respect to the convolution product,*

$$(f_1 * f_2)(g) = \int f_1(h) f_2(h^{-1}g) d\mu^{r(g)}(h)$$

and involution $f^*(g) = \overline{f(g^{-1})}$.

As with groups, $C_c(G)$ can potentially be completed with many inequivalent C^* -norms. The two most commonly used norms are the reduced norm $\|\cdot\|_r$ and the full norm $\|\cdot\|_{max}$. If G is amenable—in an appropriate sense—then these norms coincide (see Section B.3). To describe these norms, we do the usual trick of representing $C_c(G)$ as bounded operators on a Hilbert space. However, for technical reasons we restrict ourselves to I -norm bounded representations of $C_c(G)$. In all the cases we deal with, our groupoids are amenable. Hence, we omit describing the I -norm, and instead refer the reader [Ren80], [Pat99], or [Wil19].

We begin by describing the reduced norm. Suppose that G is a Haar groupoid with Haar system $\mu = \{\mu^u\}_{u \in G^{(0)}}$. For each $u \in G^{(0)}$ we define a representation $\text{Ind}_u : C_c(G) \rightarrow \mathcal{B}(L^2(G_u, \mu_u))$ via the formula

$$\text{Ind}_u(f)\xi(g) = \int_{G_u} f(h)\xi(h^{-1}g) d\mu_u(g), \quad (\text{B.1})$$

for all $f, \xi \in C_c(G)$ and $g \in G_u$. It follows from [Wil19, Proposition 1.41] that Ind_u defines an I -norm bounded representation of $C_c(G)$.

Definition B.2.4. The completion of $C_c(G)$ in the norm,

$$\|f\|_r := \sup_{u \in G^{(0)}} \|\text{Ind}_u(f)\|,$$

is denoted $C_r^*(G)$ and called the *reduced C^* -algebra of G* .

Definition B.2.5. The completion of $C_c(G)$ in the norm,

$$\|f\|_{max} := \sup_{\pi} \|\pi(f)\|,$$

where the supremum is over all I -norm bounded representations of $C_c(G)$, is denoted $C^*(G)$ and called the *full C^* -algebra of G* . Since each representation is bounded by the I -norm the supremum is finite.

As it stands, the C^* -algebras defined above depend on the choice of Haar system for G . In general, if G is a second-countable locally compact Hausdorff groupoid which is equipped with two different Haar systems, then the resulting C^* -algebras are Morita equivalent (see [Wil19, Proposition 2.74]). This is not a concern to us as we always have a concrete Haar system.

Since the regular representations Ind_u are I -norm bounded, the identity map on $C_c(G)$ extends to a surjective $*$ -homomorphism $C^*(G) \rightarrow C_r^*(G)$.

As with groups, groupoid homomorphisms do not necessarily induce maps at the level of C^* -algebras. For one, there needs to be some compatibility between the Haar systems on the groupoids and the groupoid homomorphism.

Definition B.2.6. A groupoid homomorphism $\phi : G \rightarrow H$ between locally compact Hausdorff groupoids G and H with Haar systems $\{\mu^u\}_{u \in G^{(0)}}$ and $\{\nu^v\}_{v \in H^{(0)}}$ is said to be *Haar preserving* if for all $u \in G^{(0)}$ we have $\phi_*\mu^u = \nu^{\phi(u)}$, where ϕ_* denotes the pushforward on measures.

There are two main types of homomorphisms of groupoids that we are interested in, both of which induce maps on the corresponding groupoid C^* -algebras. The first are continuous, proper, surjective, Haar-preserving groupoid homomorphisms.

Proposition B.2.7 ([AM18, Proposition 2.6 & Proposition 2.7]). *Let G and H be locally compact Hausdorff groupoids with Haar systems $\{\mu^u\}_{u \in G^{(0)}}$ and $\{\nu^v\}_{v \in H^{(0)}}$. If $\phi: G \rightarrow H$ is a continuous, proper, Haar-preserving groupoid homomorphism, then the pullback $\phi^*: C_c(H) \rightarrow C_c(G)$, given by $\phi^*(f)(g) = f(g)$, is an I -norm decreasing $*$ -homomorphism. If ϕ is also surjective, then ϕ^* induces an isometric embedding on the level of both full and reduced C^* -algebras.*

If H is an open subgroupoid of G , and $\{\mu^u\}_{u \in G^{(0)}}$ is a Haar system on G , then $\{\mu^u|_H\}_{u \in H^{(0)}}$ defines a Haar system on H . The following is well known, but we include a proof for the sake of completeness.

Proposition B.2.8. *Let G be a locally compact Hausdorff groupoid with Haar system $\{\mu^u\}_{u \in G^{(0)}}$. Suppose that H is an open subgroupoid of G with the restricted Haar system from G . Then there is a $*$ -homomorphism $i: C^*(H) \rightarrow C^*(G)$ induced by the inclusion $C_c(H) \hookrightarrow C_c(G)$.*

Proof. Let $\iota: C_c(H) \rightarrow C_c(G)$ denote $*$ -linear map given by extension by zero. As the Haar system on H is inherited from G we have, for each $f, g \in C_c(H)$

$$\iota(f) * \iota(g)(\gamma) = \int_G \iota(f)(\gamma\eta) \iota(g)(\eta^{-1}) d\mu^{s(\gamma)}(\eta).$$

This expression is non-zero only if η^{-1} and $\gamma\eta$ both belong to H , in which case $\gamma \in H$. Consequently, $\iota(f) * \iota(g) \in \iota(C_c(H))$ and $\iota(f) * \iota(g) = \iota(f * g)$.

Now, any representation $\pi: C_c(G) \rightarrow \mathcal{B}(\mathcal{H})$ of $C_c(G)$ induces a representation of $C_c(H)$. Hence, $\|f\|_G \leq \|f\|_H$ for all $f \in C_c(H)$ in the full C^* -norm. \square

A set $A \subseteq G^{(0)}$ is said to be G -invariant if for every $g \in G$ with $s(g) \in A$ we have $r(g) \in A$. If $u \in G^{(0)}$ then $(G|_A)^u$ coincides with G^u . Consequently, whenever A is an open (resp. closed) G -invariant subset of $G^{(0)}$, the restriction $G|_A$ is an open (resp. closed) subgroupoid of G which inherits the Haar system from G .

Moreover, if U is an open G -invariant set, then $C_c(G|_U)$ can be viewed as a subspace of $C_c(G)$ with the inclusion being a $*$ -homomorphism. It follows that $G^{(0)} \setminus U$ is closed and G -invariant and that restriction of functions in $C_c(G)$ to $G|_{G^{(0)} \setminus U}$ defines a surjective $*$ -homomorphism. A non-trivial consequence of this is the following.

Theorem B.2.9 ([Wil19, Theorem 5.1]). *Let G be a second-countable Haar groupoid with Haar system $\mu = \{\mu^u\}_{u \in G^{(0)}}$. Suppose that $U \subseteq G^{(0)}$ is an open G -invariant subset. Let $i: C^*(G|_U) \rightarrow C^*(G)$ be the $*$ -homomorphism induced by the inclusion $C_c(G|_U) \hookrightarrow C_c(G)$ and let $q: C^*(G) \rightarrow C^*(G|_{G^{(0)} \setminus U})$ be the $*$ -homomorphism induced by the restriction of functions $C_c(G) \twoheadrightarrow C_c(G|_{G^{(0)} \setminus U})$. Then there is a short exact sequence of full groupoid C^* -algebras*

$$0 \longrightarrow C^*(G|_U) \xrightarrow{i} C^*(G) \xrightarrow{q} C^*(G|_{G^{(0)} \setminus U}) \longrightarrow 0.$$

In particular, if the full and reduced C^ -algebras agree, then the same is true for the reduced C^* -algebras.*

B.3 | Amenability of groupoids

The concept of amenability of locally compact Hausdorff groups was defined by von Neumann during his study of the Banach-Tarski paradox [Neu29]. Originally, von Neumann formulated amenability in terms of the existence of a certain invariant measure on the group, called a mean. Since then, many equivalent characterisations of amenability have been determined. In the C^* -algebraic setting, amenability of a locally compact Hausdorff group G is equivalent to the full group C^* -algebra $C^*(G)$ and the reduced C^* -algebra $C_r^*(G)$ being isomorphic via the canonical quotient $C^*(G) \rightarrow C_r^*(G)$.

For groupoids, amenability was first formulated by Renault in his thesis [Ren80]. The groupoid case is more complicated than the group case: there are many characterisations of amenability of groupoids, but some characterisations only agree for specific classes of groupoids (see [ADR01]). Since we make limited use of amenability, we give only one characterisation. The following is a simplification of Renault's original definition [Ren80, Definition 3.1], which can be found in [Wil19, Lemma 9.5].

Definition B.3.1. A locally compact Hausdorff groupoid G with Haar system $\mu = \{\mu^u\}_{u \in G^{(0)}}$ is said to be (*topologically*) *amenable* if there exists a net $(f_\lambda)_{\lambda \in \Lambda}$ in $C_c(G)$ such that the functions $g \mapsto (f_\lambda * f_\lambda^*)(g)$ converge to the constant function 1 uniformly on every compact subset of G .

Remark B.3.2. If G is second-countable, the net in Definition B.3.1 can be taken to be a sequence.

Renault [Ren80, §3] proved that if G is an amenable groupoid with Haar system, then the full and reduced norms on $C_c(G)$ coincide. In particular, $C^*(G)$ is isomorphic to $C_r^*(G)$ (see [Wil19, Proposition 9.6]). There is another definition of amenability for groupoids which is also named topological amenability (see [Ren15, Definition 2.1]). Fortunately, if G is a second-countable, locally compact Hausdorff groupoid with Haar system, both of these notions coincide (see [Wil19, Theorem 9.43]).

Groupoid bundles are a type of groupoid for which amenability is relatively straightforward to determine.

Definition B.3.3 ([Ren15, Definition 3.3]). A locally compact groupoid G is a *groupoid bundle* over a locally compact Hausdorff space T , if there exists a continuous open surjection $p : G^{(0)} \rightarrow T$ which is invariant in the sense that $p \circ r = p \circ s$.

One advantage of a groupoid bundles, is that amenability of G can be characterised in terms of amenability of the fibres $G(t) := G|_{p^{-1}(t)}$.

Theorem B.3.4 ([Ren15, Theorem 3.5]). *Let G be a second-countable locally compact groupoid with Haar system. Suppose that G is a groupoid bundle over a locally compact Hausdorff space T with continuous open surjection $p : G^{(0)} \rightarrow T$. Then the following are equivalent:*

- (i) G is amenable;
- (ii) for all $t \in T$, $G(t)$ is amenable.

B.4 | Groupoid actions and transformation groupoids

Just as with groups, we can consider actions of groupoids on sets. A notable feature of groupoid actions is that since a groupoid G admits more than one identity element, a map is required to “anchor” the unit space of $G^{(0)}$ to the set on which G acts.

Definition B.4.1. Let G be a groupoid and X be a set with *moment map* (or *anchor map*) $m : X \rightarrow G^{(0)}$. Consider the fibre product $G \times_{s,m} X = \{(\gamma, x) \in G \times X \mid s(\gamma) = m(x)\}$. A *left action* of G on X is a map $\cdot : G \times_{s,m} X \rightarrow X$, $(\gamma, x) \mapsto \gamma \cdot x$ satisfying:

- (i) $m(x) \cdot x = x$ for all $x \in X$; and
- (ii) for all $(\gamma, \lambda) \in G^{(2)}$ and $(\lambda, x) \in G \times_{s,m} X$ we have $(\gamma\lambda) \cdot x = \gamma \cdot (\lambda \cdot x)$.

Condition (ii) implies that $m(\gamma \cdot x) = r(\gamma)$. If G is a topological groupoid and X is a topological space, then X is a *left G -space* (or just *G -space*) if $m : X \rightarrow G^{(0)}$ and $\cdot : G \times_{s,m} X \rightarrow X$ are both continuous. Our moment maps are not assumed to be open—contrary to much of the older literature—so some care is required when applying existing results from the literature.

Groupoid actions give rise to transformation groupoids.

Definition B.4.2. Suppose that G acts on a set X . Then we can form the *transformation groupoid* $G \ltimes X$. As a set $G \ltimes X$ is the fibre product

$$G \ltimes X := G \times_{s,m} X = \{(\gamma, x) \in G \times X \mid s(\gamma) = m(x)\}.$$

Multiplication is given by $(\gamma, \eta \cdot x)(\eta, x) = (\gamma\eta, x)$ for $(\gamma, \eta) \in G^{(2)}$ and inversion is given by $(\gamma, x)^{-1} = (\gamma^{-1}, \gamma \cdot x)$.

Example B.4.3. If G is a groupoid then G acts on its unit space $G^{(0)}$. The moment map is given by the source map and the action is given by $\gamma \cdot s(\gamma) = r(\gamma)$ for each $\gamma \in G$. In this case the transformation groupoid $G \ltimes G^{(0)}$ is isomorphic to G via the map $(\gamma, s(\gamma)) \mapsto \gamma$. \triangle

The unit space of $G \ltimes X$ can be identified with X via the map $x \mapsto (m_X(x), x)$. If G is a locally compact Hausdorff groupoid, and X is a locally compact Hausdorff G -space then $G \ltimes X$ is a locally compact Hausdorff groupoid. For each $x \in X$ the range fibre $(G \ltimes X)^x$ can be identified with $G^{m(x)}$. With this identification we can lift a Haar system for G to a Haar system for $G \ltimes X$.

Lemma B.4.4 ([Wil19, Ex 2.1.7 p.358]). *Let G be a locally compact Hausdorff groupoid. Suppose that X is a G -space with moment map $m : X \rightarrow G^{(0)}$, and that $\{\mu^\gamma\}_{\gamma \in G^{(0)}}$ is a Haar system on G . For each $x \in X$ and $f \in C_c(G \ltimes X)$ define ν^x by*

$$\nu_x(f) = \int_G f(\gamma, x) d\mu^{m(x)}(\gamma).$$

Then $\{\nu^x\}_{x \in X}$ defines a Haar system on $G \ltimes X$.

We also have the following amenability result.

Lemma B.4.5 ([Wil19, Corollary 9.30]). *If G is an amenable groupoid, then for every G -space X the transformation groupoid $G \ltimes X$ is amenable.*

Definition B.4.6. Let G be a topological groupoid and suppose that X and Y are both G -spaces with moment maps m_X and m_Y . A continuous map $p : X \rightarrow Y$ is said to be G -equivariant if $p(\gamma \cdot x) = \gamma \cdot p(x)$ for all $(\gamma, x) \in G \times_{s, m_X} X$. In particular, $s(\gamma) = m_Y(p(x))$ whenever $s(\gamma) = m_X(x)$.

If $p : X \rightarrow Y$ is G -equivariant, then $m_X(x) = m_Y(p(x))$ for all $x \in X$. A G -equivariant map induces a groupoid homomorphism on the corresponding transformation groupoids.

Lemma B.4.7. Suppose that $p : X \rightarrow Y$ is a G -equivariant map between locally compact Hausdorff G -spaces X and Y . Define $p^\times : G \ltimes X \rightarrow G \ltimes Y$ by

$$p^\times(\gamma, x) = (\gamma, p(x)).$$

Then p^\times is a topological groupoid homomorphism

Proof. The G -invariance of p implies that composable pairs are sent to composable pairs, and the homomorphism property follows. Continuity follows immediately from the continuity of p . \square

APPENDIX C

Topological constructions

C.1 | Fibre products

Suppose that X , Y , and Z , are topological spaces, and let $\alpha : X \rightarrow Z$ and $\beta : Y \rightarrow Z$ be continuous maps. The *fibre product* or *pullback* of X and Y with respect to the maps α and β is

$$X \times_{\alpha, \beta} Y := \{(x, y) \in X \times Y \mid \alpha(x) = \beta(y)\},$$

which we equip with the subspace topology inherited from $X \times Y$. Fibre products have the following universal property: if W is a topological space, and $\varphi : W \rightarrow X$ and $\psi : W \rightarrow Y$ are continuous maps satisfying $\alpha \circ \varphi = \beta \circ \psi$, then there is a unique continuous map $\sigma : W \rightarrow X \times_{\alpha, \beta} Y$ such that the diagram

$$\begin{array}{ccccc} W & & & & \\ & \searrow \psi & & & \\ & & X \times_{\alpha, \beta} Y & \longrightarrow & Y \\ & \swarrow \varphi & \downarrow & & \downarrow \beta \\ & & X & \xrightarrow{\alpha} & Z \end{array}$$

$\exists! \sigma$ (indicated by a dotted arrow from W to $X \times_{\alpha, \beta} Y$)

commutes. We record the following topological properties of fibre products since we refer to them frequently.

Lemma C.1.1. *Let X , Y , and Z be topological spaces and suppose that Z is Hausdorff. Let $\alpha : X \rightarrow Z$ and $\beta : Y \rightarrow Z$ be continuous maps. Let $A \subseteq X$ and $B \subseteq Y$ and consider the subset $A \times_{\alpha, \beta} B$ of $X \times_{\alpha, \beta} Y$. Then*

- (i) *if A is open in X and B is open in Y , then $A \times_{\alpha, \beta} B$ is open in $X \times_{\alpha, \beta} Y$;*
- (ii) *if A is closed in X and B is closed in Y , then $A \times_{\alpha, \beta} B$ is closed in $X \times_{\alpha, \beta} Y$; and*
- (iii) *if A is compact in X and B is compact in Y , then $A \times_{\alpha, \beta} B$ is compact in $X \times_{\alpha, \beta} Y$.*

The collection $\{A \times_{\alpha, \beta} B \mid A \subseteq X, B \subseteq Y \text{ open}\}$ is a basis for the topology on $X \times_{\alpha, \beta} Y$.

Proof. (i) and (ii) follow immediately from the fact that $A \times_{\alpha, \beta} B = (A \times B) \cap X \times_{\alpha, \beta} Y$. For (iii) it suffices to show that $X \times_{\alpha, \beta} Y$ is closed in $X \times Y$ for then $A \times_{\alpha, \beta} B$ is a closed subset of the compact set $A \times B$. Suppose that $X \times_{\alpha, \beta} Y$ is non-empty and fix a net $(x_\lambda, y_\lambda)_{\lambda \in \Lambda}$

in $X \times_{\alpha, \beta} Y$ which converges to $(x, y) \in X \times Y$. Then $\alpha(x_\lambda) \rightarrow \alpha(x)$ and $\beta(y_\lambda) \rightarrow \beta(y)$ in Z . Hausdorffness of Z now implies that $\alpha(x) = \beta(y)$ so that $(x, y) \in X \times_{\alpha, \beta} Y$. \square

C.2 | Adjunction groupoids

Suppose that X and Y are topological spaces, Z is closed in X , and that $f: Z \rightarrow Y$ is a continuous injection. Then we can form the *adjunction space*,

$$X \sqcup_f Y := X \sqcup Y / \{x \sim f(x) \mid x \in Z\},$$

which we equip with the quotient topology. Adjunction spaces can be defined without the assumption of injectivity on f (see for example [Eng89, p.93]), but we do not require this generality. As a set $X \sqcup_f Y$ can be identified with $(X \setminus Z) \sqcup Y$. Since we stipulate that f is injective, $X \sqcup_f Y$ can also be identified with $X \sqcup (Y \setminus f(Z))$. These identifications do not respect the topology as $X \sqcup_f Y$ does not typically carry the disjoint-union topology.

Let $q: X \sqcup Y \rightarrow X \sqcup_f Y$ denote the quotient map and let $i: X \rightarrow X \sqcup_f Y$ and $j: Y \rightarrow X \sqcup_f Y$ denote the inclusions of X and Y into $X \sqcup Y$, respectively, composed with q . Note that j is injective, and injectivity of f implies that i is also injective.

Adjunction spaces admit the following universal property. If W is a topological space, and $\varphi: X \rightarrow W$ and $\psi: Y \rightarrow W$ are continuous maps such that $\varphi(x) = \psi(f(x))$ for all $x \in Z$, then there exists a unique continuous map $\sigma: X \sqcup_f Y \rightarrow W$ such that the diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & Y \\ \downarrow & & \downarrow j \\ X & \xrightarrow{i} & X \sqcup_f Y \end{array} \quad \begin{array}{c} \searrow \psi \\ \downarrow \exists! \sigma \\ W \end{array}$$

(A curved arrow φ goes from X to W .)

commutes. This follows by combining the universal property of $X \sqcup Y$ with the universal property of the quotient.

Lemma C.2.1. *Let X and Y be topological spaces, Z a closed subset of X , and let $f: Z \rightarrow Y$ be a continuous injection. The adjunction space $X \sqcup_f Y$ has the following properties:*

- (i) *the map $j: Y \rightarrow X \sqcup_f Y$ is closed, hence a homeomorphism onto its image;*
- (ii) *if $f: Z \rightarrow Y$ is open then so is $i: X \rightarrow X \sqcup_f Y$, in which case i is a homeomorphism onto its image;*
- (iii) *if X and Y are normal Hausdorff spaces then so is $X \sqcup_f Y$.*

Proof. For (i) note that if C is closed in Y then $q^{-1}(j(C)) = f^{-1}(C) \sqcup C$, which is closed in $X \sqcup Y$ since Z is closed in X . Hence, $j(C)$ is closed in the quotient topology. For (ii) suppose that $f: Z \rightarrow Y$ is open and U is open in X . Since $U \cap Z$ is open in Z , $q^{-1}(i(U)) = U \sqcup f(U \cap Z)$ is open in $X \sqcup Y$. Hence $i(U)$ is open in the quotient topology.

For (iii) suppose that C and D are disjoint closed subsets of $X \sqcup_f Y$. Since Y is normal Hausdorff, Urysohn's Lemma ([Rud87, Theorem 2.12]) gives a continuous function $\psi: Y \rightarrow [0, 1]$ such that $\psi(j^{-1}(C \cap Y)) = \{0\}$ and $\psi(j^{-1}(D \cap Y)) = \{1\}$. Since X is normal, the Tietze Extension Theorem ([Rud87, Theorem 20.4]) gives a continuous map $\varphi: X \rightarrow [0, 1]$ which takes the value 0 on $i^{-1}(C)$, the value 1 on $i^{-1}(D)$, and extends the map $(\psi \circ f)|_Z: Z \rightarrow [0, 1]$. It now follows from the universal property of adjunction spaces that there exists a continuous function $\sigma: X \sqcup_f Y \rightarrow [0, 1]$ such that $\sigma(C) = \{0\}$ and $\sigma(D) = \{1\}$. Since C and D can be separated by a continuous function we see that $X \sqcup_f Y$ is normal Hausdorff. \square

Although the adjunction space of two normal spaces is again normal, it is possible for X and Y to be locally compact, second-countable, and Hausdorff, with $X \sqcup_f Y$ not locally compact. This can be seen in the following examples.

Example C.2.2 (cf. [Eng89, Example 1.6.19]). Let $Y = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$, equipped with the subspace topology from \mathbb{R} , and let $X = Y \times \mathbb{N}$. We can identify Y with the one-point compactification of \mathbb{N} . Both X and Y are second-countable, locally compact, Hausdorff spaces. Consider the closed subspace $Z = \{0\} \times \mathbb{N}$ of X and define $f: Z \rightarrow Y$ by $f(0, n) = 1/n$. Then f is continuous, injective, and open.

We claim that $X \sqcup_f Y$ is not locally compact. Let $i: X \rightarrow X \sqcup_f Y$ and $j: Y \rightarrow X \sqcup_f Y$ denote the canonical inclusions and let $q: X \sqcup Y \rightarrow X \sqcup_f Y$ be the quotient map. Suppose that U is an open neighbourhood of $j(0)$. Then $j^{-1}(U)$ is open and contains 0. Hence, there exist $N \in \mathbb{N}$ such that $\{1/n \mid n \geq N\} \subseteq j^{-1}(U)$. Since $i^{-1}(U)$ is open we can find a basic open set $W \times V \subseteq i^{-1}(U)$ with W open in Y and V open in \mathbb{N} . Since $i^{-1}(U)$ contains $(0, n)$ for all $n \geq N$ we can assume that there exists $M \in \mathbb{N}$ and $K \geq N$ such that $W = \{0\} \cup \{1/m \mid m \geq M\}$ and $V = \{k \mid k \geq K\}$. Consequently, $i^{-1}(U)$ contains a sequence of the form $((m, k))_{k \geq K}$ with $m \neq 0$. It follows that $(q(m, k))_{k \geq K}$ is a sequence in U with no limit points. Hence, $X \sqcup_f Y$ is not locally compact. \triangle

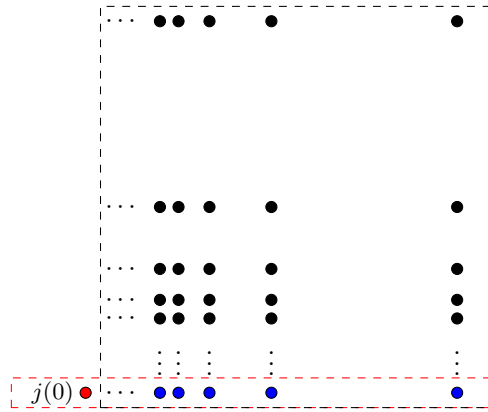


Figure C.1: The space $X \sqcup_f Y$ from Example C.2.2. The set $i(Z) = j(f(Z))$ is highlighted in blue. The red dashed rectangle indicates $j(Y)$ while the black dashed rectangle indicates $i(X)$.

Example C.2.3. Suppose that $X = \mathbb{R}^2$, $Z = \{0\} \times \mathbb{R}$, $Y = \mathbb{T}$, and $f: Z \rightarrow \mathbb{T}$ corresponds to the one point compactification of \mathbb{R} . Then f is a continuous, open, injection. Let ∞ denote the unique point in $\mathbb{T} \setminus f(Z)$. We use a similar argument to Example C.2.2 to see that $X \sqcup_f Y$ is not locally compact.

Suppose that U is an open neighbourhood of $j(\infty)$. Since $i^{-1}(U)$ is open, there exists open intervals W and V in \mathbb{R} such that $W \times V \subseteq i^{-1}(U)$. Since $i^{-1}(U)$ is not bounded,

we can take $V = (a, \infty)$ for some $a \in \mathbb{R}$. Let $b \in W$ with $b \neq 0$ and for each $n \in \mathbb{N}$ with $n \geq a$ let $x_n = (b, n) \in W \times V$. Then $(q(x_n))_n$ is a sequence in U with no limit points. Hence, $X \sqcup_f Y$ is not locally compact. \triangleleft

One way to ensure that $X \sqcup_f Y$ is locally compact is to impose the fairly strong hypothesis that Z is clopen in X .

Lemma C.2.4. *Let X, Y, Z , and f be as in Lemma C.2.1. If Z is clopen in X then $X \sqcup_f Y$ is homeomorphic to $(X \setminus Z) \sqcup Y$ with the disjoint union topology. In particular, $j: Y \rightarrow X \sqcup_f Y$ is an open map. Moreover, if X and Y are both second-countable, locally compact, or Hausdorff, then so is $X \sqcup_f Y$.*

Proof. Since Z is clopen in X , X is homeomorphic to $(X \setminus Z) \sqcup Z$ with the disjoint union topology. It follows that $X \sqcup_f Y$ is homeomorphic to $(X \setminus Z) \sqcup (Z \sqcup_f Y)$ with the disjoint union topology. Since $Z \sqcup_f Y \simeq Y$ the result follows. \square

Lemma C.2.5. *If Z is clopen in X and $f: Z \rightarrow Y$ is open then the quotient map $q: X \sqcup Y \rightarrow X \sqcup_f Y$ is open.*

Proof. It follows from Lemma C.2.1 and Lemma C.2.4 that $i: X \rightarrow X \sqcup_f Y$ and $j: Y \rightarrow X \sqcup_f Y$ are open. Each open set $U \subseteq X \sqcup Y$ can be written as a disjoint union $V_X \sqcup V_Y$ with V_X open in X and V_Y open in Y . Hence, $q(U) = q(V_X \cup V_Y) = i(V_X) \cup j(V_Y)$ is open in $X \sqcup_f Y$. \square

Although in the main body of this thesis we deal primarily with the case where Z is clopen in X , this condition can be relaxed substantially. To do this we introduce the notion of bi-quotients.

Definition C.2.6 ([Mic72, Definition 3.1]). Let X and Y be topological spaces. A quotient map $f: X \rightarrow Y$ is said to be a *bi-quotient* if for every $y \in Y$ and every open cover \mathcal{U} of $f^{-1}(y)$ there exists finitely many $U \in \mathcal{U}$ such that the sets $f(U)$ cover a neighbourhood of y .

Open quotient maps and perfect quotient maps are both examples of bi-quotients. Bi-quotients behave well with respect to local compactness as the following result of Michael shows.

Theorem C.2.7 ([Mic72, Theorem 3.A.1, §3C]). *Let Y be a topological space. Then:*

- (i) *if Y is Hausdorff, then Y is locally metrisable and locally compact if and only if Y is the image of a metrisable, locally compact space under a bi-quotient map;*
- (ii) *if Y is T_0 , then Y is second-countable if and only if it is the image of a separable metrisable space under a bi-quotient map.*

Corollary C.2.8. *Let X and Y be second-countable locally compact Hausdorff spaces. Suppose that Z is a closed subset of X , and let $f: Z \rightarrow Y$ be a continuous injection. Suppose that X and Y are second-countable locally compact Hausdorff spaces. If the quotient map $q: X \sqcup Y \rightarrow X \sqcup_f Y$ is a bi-quotient, then $X \sqcup_f Y$ is a second-countable locally compact Hausdorff space.*

Proof. Since second-countable locally compact Hausdorff spaces are normal, it follows from (iii) of Lemma C.2.1 that $X \sqcup_f Y$ is normal Hausdorff. Second-countable locally compact Hausdorff spaces are metrisable by Urysohn's Metrisation Theorem, so Theorem C.2.7 implies that $X \sqcup_f Y$ is locally compact and second-countable. \square

Checking whether a map is a bi-quotient can be difficult in general. For our application of adjunction spaces we give a checkable condition that implies $q: X \sqcup Y \rightarrow X \sqcup_f Y$ is a bi-quotient.

Lemma C.2.9. *Suppose that X, Y, Z, f are as in Corollary C.2.1 and suppose that $f: Z \rightarrow Y$ is open. Suppose that for all $y \in Y \setminus f(Z)$ there exists an open neighbourhood V of y in Y such that $f^{-1}(V)$ is open (possibly empty) in X . Then $q: X \sqcup Y \rightarrow X \sqcup_f Y$ is a bi-quotient. Moreover, for such a neighbourhood V , $q(V) = j(V)$ is open in $X \sqcup_f Y$.*

Proof. Suppose that $z \in X \sqcup_f Y$. First consider the case where $z \in j(Y \setminus f(Z))$. Then $q^{-1}(z) = \{y\}$ for some $y \in Y \setminus f(Z)$. Fix an open neighbourhood U of y and suppose that V is an open neighbourhood of y such that $f^{-1}(V)$ is open in X . Then $f^{-1}(V \cap U) = f^{-1}(V) \cap f^{-1}(U)$ is open in $f^{-1}(V) \subseteq Z \subseteq X$. Since $f^{-1}(V)$ is also open in X it follows that $f^{-1}(V \cap U)$ is open in X .

It now follows that $q(V \cap U)$ is open in $X \sqcup_f Y$ because $q^{-1} \circ q(V \cap U) = f^{-1}(V \cap U) \sqcup (V \cap U)$ is open in $X \sqcup Y$. Since $q(V \cap U) \subseteq q(U)$ is an open neighbourhood of z we have verified the bi-quotient condition for $z \in j(Y \setminus f(Z))$.

Now suppose that $z \in i(X)$. Then there exists a unique $x \in X$ such that $q(x) = z$. Take an open cover \mathcal{U} of $q^{-1}(z)$. Choose $U \in \mathcal{U}$ such that $x \in U$. Since $i: X \rightarrow X \sqcup_f Y$ is an open map it follows that $i(U)$ is an open neighbourhood of z ; verifying the bi-quotient condition for $z \in i(X)$.

The final statement follows because $q^{-1}(q(V)) = f^{-1}(V) \sqcup V$ is open in $X \sqcup Y$. \square

Recall the basic notions of topological groupoids from Appendix B. If X and Y are topological groupoids, Z is a closed subgroupoid of X , and $f: Z \rightarrow Y$ is an injective continuous groupoid homomorphism, then one can ask whether $X \sqcup_f Y$ admits the structure of a topological groupoid.

As it turns out, with the correct hypotheses $X \sqcup_f Y$ can be equipped with a topological groupoid structure, and the inclusions $i: X \rightarrow X \sqcup_f Y$ and $j: Y \rightarrow X \sqcup_f Y$ become continuous groupoid homomorphisms. As far as the author is aware, this construction is original, though we note that there is some resemblance to the factor groupoids considered by Putnam in [Put98].

Theorem C.2.10 (Adjunction groupoids). *Let X and Y be second-countable locally compact Hausdorff groupoids. Let Z be a closed subgroupoid of X and suppose that $f: Z \rightarrow Y$ is a continuous, open, injective, groupoid homomorphism. Suppose also that the hypotheses of Lemma C.2.9 are satisfied. Finally, suppose that $f(Z^{(0)})$ is Y -invariant (so $f(Z) = Y|_{f(Z^{(0)})}$). Then $G := X \sqcup_f Y$ is a second-countable locally compact Hausdorff groupoid with composable pairs*

$$G^{(2)} = \{(q(\gamma), q(\eta)) \mid \gamma, \eta \in X \sqcup Y \text{ and } q(s(\gamma)) = q(r(\eta))\},$$

multiplication given for $\gamma, \eta \in X \sqcup Y$ by

$$q(\gamma)q(\eta) = \begin{cases} i(\gamma\eta) & \text{if } \gamma, \eta \in X, \\ j(\gamma\eta) & \text{if } \gamma, \eta \in Y, \\ i(\gamma f^{-1}(\eta)) & \text{if } \gamma \in X, \eta \in Y, \text{ and} \\ i(f^{-1}(\gamma)\eta) & \text{if } \gamma \in Y, \eta \in X, \end{cases}$$

and inversion given by $q(\gamma)^{-1} = q(\gamma^{-1})$. The range and source maps satisfy

$$r(q(\gamma)) = q(r(\gamma)) \quad \text{and} \quad s(q(\gamma)) = q(s(\gamma)).$$

Both $i: X \rightarrow G$ and $j: Y \rightarrow G$ are groupoid homomorphisms, the unit space is given by $G^{(0)} = i(X^{(0)}) \cup j(Y^{(0)})$, and $i(X^{(0)})$ is an open G -invariant subset of $G^{(0)}$. If the source maps (equivalently the range maps) are open for X and Y then the source map (equivalently the range map) is open for G . If X and Y are étale, then so is G .

Proof. We begin by showing that the formula for multiplication in G makes sense when one element belongs to $i(X)$ and the other to $j(Y)$. So suppose that $\gamma \in X$ and $\eta \in Y$ satisfy $q(s(\gamma)) = q(r(\eta))$. Then $s(\gamma) \in Z$, $r(\eta) \in f(Z)$ and $f(s(\gamma)) = r(\eta)$. Since $f(Z) = Y|_{f(Z)^{(0)}}$ and $f(Z)^{(0)}$ is Y -invariant it follows that $\eta \in f(Z)$. As f is an injective groupoid homomorphism, $r(f^{-1}(\eta)) = f^{-1}(r(\eta))$. Consequently, $(\gamma, f^{-1}(\eta)) \in X^{(2)}$ and so the product is well-defined. A symmetric argument holds if $\gamma \in Y$ and $\eta \in X$. If $x \in X$ and $\eta \in Z$ then $q(\gamma)q(\eta) = q(\gamma)q(f(\eta))$. Hence, G is a groupoid.

It follows that $i: X \rightarrow G$ and $j: Y \rightarrow G$ are groupoid homomorphisms, and hence isomorphisms onto their images. In particular, $i(X^{(0)}) \cup j(Y^{(0)}) \subseteq G^{(0)}$. The reverse inclusion is easily observed from the definition of multiplication in G . The range and source maps on G are unambiguously given by $r(q(\gamma)) = q(r(\gamma))$ and $s(q(\gamma)) = q(s(\gamma))$ for γ in either X or Y .

The Y -invariance of $f(Z^{(0)})$ implies that $i(X^{(0)})$ is G -invariant: if $s(q(\gamma)) \in i(X^{(0)})$ then either $\gamma \in X$ or $\gamma \in f(Z)$. In the latter case Y -invariance of $f(Z^{(0)})$ implies $r(q(\gamma)) \in j(f(Z^{(0)})) \subseteq i(X^{(0)})$. Accordingly, $G^{(0)} \setminus i(X^{(0)})$ is also G -invariant and $G \setminus i(X)$ is isomorphic to $Y \setminus f(Z)$.

We now deal with the topology. Lemma C.2.9 and Corollary C.2.8 show that $X \sqcup_f Y$ is a second-countable locally compact Hausdorff space. Lemma C.2.1 implies that $i(X)$ is open and $j(Y)$ is closed in G .

We claim that multiplication in G is continuous. Suppose that $(x_\lambda, y_\lambda) \rightarrow (x, y)$ in $G^{(2)}$. Then $x_\lambda \rightarrow x$ and $y_\lambda \rightarrow y$ in G . First suppose that $xy \in i(X)$. Since $i(X) = G|_{i(X^{(0)})}$ it follows that we have $x, y \in i(X)$. As $i(X)$ is open in G , there exists λ_0 such that $\lambda \geq \lambda_0$ implies $x_\lambda, y_\lambda \in i(X)$. Since i is both a homeomorphism onto its image and a groupoid homomorphism, it follows from continuity of multiplication in X that $x_\lambda y_\lambda \rightarrow xy$ in G .

Now suppose that $xy \in G \setminus i(X) = j(Y \setminus f(Z))$. Since $G \setminus i(X) = G|_{G^{(0)} \setminus i(X^{(0)})}$ is G -invariant, $x, y \in G \setminus i(X)$. It follows by assumption and Lemma C.2.9 that we can find open sets V_x and V_y in Y such that $q(V_x)$ and $q(V_y)$ are open neighbourhoods of x and y , respectively. Hence, there exists λ_0 such that $\lambda \geq \lambda_0$ implies $x_\lambda \in q(V_x)$ and $y_\lambda \in q(V_y)$. In particular, the nets (x_λ) and (y_λ) are eventually in $j(Y)$ so that $(x_\lambda, y_\lambda) \in j(Y)^{(2)}$. Since j is both a homeomorphism onto its image and a groupoid homomorphism, it now follows from continuity of multiplication in Y that $x_\lambda y_\lambda \rightarrow xy$ in G . Hence multiplication in G is continuous. A similar argument shows that inversion is continuous, and so G is a topological groupoid.

Since $i: X \rightarrow G$ is open and injective, the restriction $i|_{X^{(0)}}: X^{(0)} \rightarrow G^{(0)}$ is an open map. In particular, $i(X^{(0)})$ is open in $G^{(0)}$.

Now suppose that $s_X: X \rightarrow X^{(0)}$ and $s_Y: Y \rightarrow Y^{(0)}$ are open. To see that $s: G \rightarrow$

$G^{(0)}$ is open, let U be an open subset of G . Then,

$$\begin{aligned}
 s(U) &= s(q(q^{-1}(U))) \\
 &= s(q((q^{-1}(U) \cap X) \cup (q^{-1}(U) \cap Y))) \\
 &= s(q(q^{-1}(U) \cap X)) \cup s(q(q^{-1}(U) \cap Y)) \\
 &= q(s_X(q^{-1}(U) \cap X)) \cup q(s_Y(q^{-1}(U) \cap Y)) \\
 &= i(s_X(i^{-1}(U))) \cup j(s_Y(j^{-1}(U))).
 \end{aligned}$$

Since both $i|_{X^{(0)}}$ and s_X are open $i(s_X(i^{-1}(U)))$ is open in $G^{(0)}$. On the other hand,

$$s_Y(j^{-1}(U)) = s_Y(j^{-1}(U) \cap f(Z)) \cup s_Y(j^{-1}(U) \cap (Y \setminus f(Z))).$$

Since $f: Z \rightarrow Y$ is an injective groupoid homomorphism and $i|_Z = j \circ f$ we have

$$\begin{aligned}
 j(s_Y(j^{-1}(U) \cap f(Z))) &= i(f^{-1}(s_Y(j^{-1}(U) \cap f(Z)))) \\
 &= i(s_X(f^{-1}(j^{-1}(U) \cap f(Z)))) \\
 &= i(s_X(i^{-1}(U) \cap Z)) \\
 &\subseteq i(s_X(i^{-1}(U))).
 \end{aligned}$$

Now, for each $y \in j^{-1}(U) \cap (Y \setminus f(Z))$ let V_y an open neighbourhood of y such that $f^{-1}(V_y)$ is open in X . Since s_Y is an open map $W_y := V_y \cap s_Y(j^{-1}(U))$ is open in $Y^{(0)}$. The set $f^{-1}(W_y) = f^{-1}(V_y) \cap f^{-1}(s_Y(j^{-1}(U))) = f^{-1}(V_y) \cap s_X(i^{-1}(U))$ is open in $X^{(0)}$. Consequently, $j(W_y)$ is open in $G^{(0)}$. Putting everything together,

$$\begin{aligned}
 s(U) &= i(s_X(i^{-1}(U))) \cup j(s_Y(j^{-1}(U))) \\
 &= i(s_X(i^{-1}(U))) \cup j(s_Y(j^{-1}(U) \cap f(Z))) \cup \bigcup_{y \in j^{-1}(U) \cap (Y \setminus f(Z))} j(W_y) \\
 &= i(s_X(i^{-1}(U))) \cup \bigcup_{y \in j^{-1}(U) \cap (Y \setminus f(Z))} j(W_y)
 \end{aligned}$$

is open in $G^{(0)}$. Thus, $s: G \rightarrow G^{(0)}$ is an open map.

Finally, suppose that X and Y are both étale. Since X is étale, $i(X^{(0)})$ is open in G . By assumption, for each $z \in j(Y \setminus f(Z))$ we can find an open subset V_z of Y such that $j(V_z)$ is an open neighbourhood of z in G . Then $Y^{(0)} \cap V_z$ is open in Y because Y is étale. Since j is injective, $j(Y^{(0)} \cap V_z) = j(Y^{(0)}) \cap j(V_z) = G^{(0)} \cap j(V_z)$. So

$$j(Y^{(0)} \cap V_z) \cap i(X^{(0)}) = j(V_z) \cap i(X^{(0)})$$

is open in G . It follows that

$$q^{-1}(Y^{(0)} \cap V_z) = i^{-1}(j(V_z) \cap i(X^{(0)})) \sqcup (Y^{(0)} \cap V_z)$$

is open in $X \sqcup Y$, so $j(Y^{(0)} \cap V_z)$ is open in G . Putting everything together we see that

$G^{(0)} = i(X^{(0)}) \cup \bigcup_{z \in Y^{(0)} \setminus f(Z^{(0)})} j(V_z \cap Y^{(0)})$ is open in G . Since the source map in G is open whenever the source maps in X and Y are open, it follows that G is étale. \square

Remark C.2.11. Although we used Lemma C.2.9 throughout the proof of Theorem C.2.10, it is entirely possible that the statement of Theorem C.2.10 is true so long as $q: X \sqcup Y \rightarrow X \sqcup_f Y$ is a bi-quotient. This generality is beyond our present needs.

Remark C.2.12. If we identify $G = X \sqcup_f Y$ with $X \sqcup (Y \setminus f(Z))$ as sets, then $G^{(2)}$ can be identified with $X^{(2)} \sqcup (Y \setminus f(Z))^{(2)}$ and the product written as

$$\gamma\eta = \begin{cases} \gamma\eta & \text{if } (\gamma, \eta) \in X^{(2)}, \\ \gamma\eta & \text{if } (\gamma, \eta) \in (Y \setminus f(Z))^{(2)}. \end{cases}$$

Remark C.2.13. Suppose that X and Y are non-étale groupoids with Haar systems $\lambda = \{\lambda_x\}_{x \in X^{(0)}}$ and $\mu = \{\mu_y\}_{y \in Y^{(0)}}$. It is not entirely clear how to “glue” λ and μ together to form a Haar system on $X \sqcup_f Y$. Such a Haar system ought to restrict to a Haar system equivalent to λ on X , and equivalent to μ on Y . This would require, at the least, equivalence of the restrictions of λ to Z and of μ to $f(Z)$.

A solution to this problem is needed to construct a groupoid model for topological quivers algebras using the strategy outlined in Chapter 3. Accordingly, it merits further investigation, which will be the subject of future work.

In the étale setting, amenability of X and Y implies amenability of $X \sqcup_f Y$.

Corollary C.2.14. *Suppose that X, Y , and $f: Z \rightarrow Y$ satisfy the hypotheses of Theorem C.2.10 and that X and Y are étale. If X and Y are amenable, then so is $G := X \sqcup_f Y$.*

Proof. Theorem C.2.10 shows that $i(X^{(0)})$ is an open $X \sqcup_f Y$ -invariant subset of $G^{(0)}$. So, $G|_{i(X^{(0)})} = i(X)$ and $G|_{G^{(0)} \setminus i(X^{(0)})} = j(X \setminus f(Z))$. Since Y is amenable [Wil19, Proposition 9.83] implies $Y \setminus f(Z)$ is amenable, and since X is also amenable it follows from [Wil19, Proposition 9.83] that $X \sqcup_f Y$ is amenable. \square

Corollary C.2.15. *Suppose that X, Y and, $f: Z \rightarrow Y$ satisfy the hypotheses of Theorem C.2.10 and that X and Y are étale. Let $i_*: C^*(X) \rightarrow C^*(X \sqcup_f Y)$ be the $*$ -homomorphism induced by Proposition B.2.8. Then there is a short exact sequence*

$$0 \longrightarrow C^*(X) \xrightarrow{i_*} C^*(X \sqcup_f Y) \longrightarrow C^*(Y \setminus f(Z)) \longrightarrow 0$$

of full groupoid C^ -algebras. If Z is also open in X , we have $*$ -homomorphisms $f_*: C^*(Z) \rightarrow C^*(Y)$, $j_*: C^*(Y) \rightarrow C^*(X \sqcup_f Y)$, and $\iota_*: C^*(Z) \rightarrow C^*(X)$ induced Proposition B.2.8. Then the diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^*(Z) & \xrightarrow{f_*} & C^*(Y) & \longrightarrow & C^*(Y \setminus f(Z)) \longrightarrow 0 \\ & & \downarrow \iota_* & & \downarrow j_* & & \parallel \\ 0 & \longrightarrow & C^*(X) & \xrightarrow{i_*} & C^*(X \sqcup_f Y) & \longrightarrow & C^*(Y \setminus f(Z)) \longrightarrow 0. \end{array} \quad (\text{C.1})$$

commutes and has exact rows.

Proof. It follows from Theorem C.2.10 that $f(Z)^{(0)}$ is a Y -invariant open subset of $Y^{(0)}$ and $i(X)^{(0)}$ is an $X \sqcup_f Y$ -invariant open subset of $(X \sqcup_f Y)^{(0)}$. We also have $j(Y \setminus f(Z)) \cong Y \setminus f(Z)$. The rows of (C.1) and their exactness now follow from Theorem B.2.9. It is straightforward to verify that the diagram commutes. \square

In light of Corollary C.2.15 it is worth recording how elements in the image of i_* and j_* multiply. If $\xi \in C_c(X)$ and $\eta \in C_c(Y)$ then,

$$i_*(\xi) * j_*(\eta)(x) = \sum_{s(y)=s(x)} i_*(\xi)(xy^{-1})j_*(\eta)(y).$$

The summands are zero unless $xy^{-1} \in i(X)$ and $y \in j(Y)$. This only occurs if $r(y) \in i(Z) = j(f(Z))$, in which case $x = xy^{-1}y \in i(X)$. Since $y \in j(Y)$, then $r(y) \in j(f(Z))$ implies $y \in j(f(Z)) = i(Z)$ by Y -invariance of $f(Z)^{(0)}$. We can therefore rewrite the product as,

$$i_*(\xi) * j_*(\eta)(x) = \begin{cases} \sum_{y \in Z_{s(w)}} \xi(wy^{-1})\eta(f(y)) & \text{if } x = i(w), \\ 0 & \text{if } x \in j(Y \setminus f(Z)). \end{cases}$$

C.3 | Perfections of continuous maps

In this section we describe a way to extend continuous maps between locally compact Hausdorff spaces to perfect maps. To begin, we recall the notion of a perfect map.

Definition C.3.1 ([Eng89, §3.7]). A map $p: X \rightarrow Y$ between topological spaces is said to be *perfect* if it is continuous, closed, surjective, and for every $y \in Y$ the set $p^{-1}(y)$ is compact in X . When Y is locally compact and Hausdorff, perfect maps coincide with continuous, proper (the preimage of compact sets is compact), surjections. Sometimes in the literature the surjectivity condition on perfect maps is dropped, and consequently in the Hausdorff case perfect maps and proper maps agree. We always require perfect maps to be surjective.

Perfect maps behave well with respect to local compactness; for example the image of a locally compact space under a perfect map is again locally compact [Mic72]. Recall that the category of locally compact Hausdorff spaces with proper continuous maps is contravariantly equivalent (via Gelfand duality) to the category of commutative C^* -algebras with non-degenerate $*$ -homomorphisms. It follows that the category of locally compact Hausdorff spaces with perfect maps is equivalent to the category of commutative C^* -algebras with non-degenerate injective $*$ -homomorphisms.

From now on we assume that X and Y are both locally compact and Hausdorff. In this section we introduce a way to extend the domain of a continuous map $p: X \rightarrow Y$ to arrive at a perfect map $\tilde{p}: X_p^+ \rightarrow Y$. We call this process *perfection*.

Constructions similar to perfections have been studied in the literature. The first was introduced by Whyburn in [Why53; Why66]; he called these compactifications of mappings. Whyburn introduced what he called a *unified space*: given $p: X \rightarrow Y$ the unified space is $X \sqcup Y$ under a suitable topology and comes with a continuous extension $\tilde{p}: X \sqcup Y \rightarrow Y$ of p such that \tilde{p} is a perfect map. The notion of *minimal perfection* we

introduce is a certain closed subset of Whyburn's unified space. Further exposition on compactifications of mappings can be found in [Cai69; DS73; Fir74; Jam89].

More recently—in the case where p is surjective—Anantharaman-Delaroche introduced the concept of a fibrewise compactification in [AD14] to study exactness of groupoids. The perfection we introduce are fibrewise compactifications in the sense of [AD14] when p is already surjective. In [Jam89, § II.8] fibrewise compactifications without the assumption of p being surjective are considered, but the resulting extension of p is not necessarily surjective.

C.3.1 The unified space

For our first example of a perfection we introduce the unified space. Following Whyburn [Why66], suppose that $p : X \rightarrow Y$ is a continuous map between Hausdorff spaces X and Y . Let $\widetilde{X}_p = X \sqcup Y$ as set and consider the map $\widetilde{p} : \widetilde{X}_p \rightarrow Y$ given by

$$\widetilde{p}(x) = \begin{cases} p(x) & \text{if } x \in X, \\ x & \text{if } x \in Y. \end{cases}$$

We equip \widetilde{X}_p with the topology generated by the basis

$$\mathcal{B} := \{U \mid U \text{ open in } X\} \cup \{\widetilde{p}^{-1}(V) \cap (\widetilde{X}_p \setminus K) \mid V \text{ is open in } Y \text{ and } K \subseteq X \text{ is compact}\}.$$

To see that \mathcal{B} is indeed a basis let U be open in X , V be open in Y , and suppose that K is compact in X . Then $U \cap \widetilde{p}^{-1}(V) \cap (\widetilde{X}_p \setminus K) = U \cap p^{-1}(V) \cap (X \setminus K)$ is open in X and therefore an element of \mathcal{B} . The other possible intersections of basic open sets clearly lie back in \mathcal{B} .

Definition C.3.2 ([Why66, §3]). We call the pair $(\widetilde{X}_p, \widetilde{p})$ the *unified space* of $p : X \rightarrow Y$.

If Y is locally compact then the topology on the unified space agrees with the one from [Why66].

Lemma C.3.3. *Let $p : X \rightarrow Y$ be a continuous map between Hausdorff spaces. If Y is locally compact, then a set $W \subseteq \widetilde{X}_p$ is open if and only if*

- (i) $W \cap X$ is open in X and $W \cap Y$ is open in Y ; and
- (ii) for any compact set $K \subseteq W \cap Y$, the set $p^{-1}(K) \cap (X \setminus W)$ is compact in X .

Proof. It is shown in [Why53] that the conditions (i) and (ii) define a topology on \widetilde{X}_p . We claim that each set in \mathcal{B} satisfies (i) and (ii).

If U is open in X then (i) and (ii) are trivially satisfied. Now suppose that V is open in Y and C is compact in X . Then $\widetilde{p}^{-1}(V) \cap (\widetilde{X}_p \setminus C) \cap X = p^{-1}(V) \cap (X \setminus C)$ is open in X and $\widetilde{p}^{-1}(V) \cap (\widetilde{X}_p \setminus C) \cap Y = V$ is open in Y . If $K \subseteq V = \widetilde{p}^{-1}(V) \cap Y$ is compact, then

$$\begin{aligned} p^{-1}(K) \cap (X \setminus (\widetilde{p}^{-1}(V) \cap (\widetilde{X}_p \setminus C))) &= (p^{-1}(K) \cap (X \setminus p^{-1}(V))) \cup (p^{-1}(K) \cap C) \\ &= p^{-1}(K) \cap C \end{aligned}$$

is compact as it is closed in C . Thus, $\tilde{p}^{-1}(V) \cap (\tilde{X}_p \setminus C)$ satisfies both (i) and (ii).

Now suppose that $W \subseteq \tilde{X}_p$ satisfies both (i) and (ii) and fix $x \in W$. We claim that there exists $V \in \mathcal{B}$ such that $x \in V \subseteq W$. If $x \in W \cap X$, then since $W \cap X \in \mathcal{B}$ we are done. On the other hand, suppose that $x \in W \cap Y$. Since $W \cap Y$ is open in Y , and Y is locally compact, there exists a compact neighbourhood $K \subseteq W \cap Y$ of x . Moreover, there exists an open neighbourhood U of x satisfying $x \in U \subseteq K$. Condition (ii) implies that $p^{-1}(K) \cap (X \setminus W)$ is compact in X . Using the fact $U \subseteq K \subseteq W \cap Y$,

$$\begin{aligned} & \tilde{p}^{-1}(U) \cap (\tilde{X}_p \setminus (p^{-1}(K) \cap (X \setminus W))) \\ &= \tilde{p}^{-1}(U) \cap ((\tilde{X}_p \setminus p^{-1}(K)) \cup (\tilde{X}_p \cap (X \setminus W))) \\ &= (\tilde{p}^{-1}(U) \cap (\tilde{X}_p \setminus p^{-1}(K))) \cup (\tilde{p}^{-1}(U) \cap W) \cup (\tilde{p}^{-1}(U) \cap Y) \\ &= U \cup (\tilde{p}^{-1}(U) \cap W) \cup U \\ &\subseteq W. \end{aligned}$$

Since $x \in \tilde{p}^{-1}(U) \cap (\tilde{X}_p \setminus (p^{-1}(K) \cap (X \setminus W))) \in \mathcal{B}$ we have found our desired neighbourhood of x . \square

Remark C.3.4. Under the hypotheses of Lemma C.3.3 a set $C \subseteq \tilde{X}_p$ is closed if and only if

- (i) $X \cap C$ is closed in X and $Y \cap C$ is closed in Y ; and
- (ii) for any compact set $K \subseteq Y \setminus C$ the set $p^{-1}(K) \cap C$ is compact.

Remark C.3.5. Under the hypotheses of Lemma C.3.3 we can characterise when a net $(x_\lambda)_{\lambda \in \Lambda}$ in \tilde{X}_p converges to $x \in \tilde{X}_p$. This characterisation depends on whether $x \in X$ or $x \in Y$.

- (i) Suppose $x \in X$. Then $x_\lambda \rightarrow x$ in \tilde{X}_p if and only if there exists $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in X$ and $x_\lambda \rightarrow x$ in X .
- (ii) Suppose $x \in Y$. Then $x_\lambda \rightarrow x$ if and only if $\tilde{p}(x_\lambda) \rightarrow \tilde{p}(x)$ and for any compact set $K \subseteq X$ there exists $\lambda_K \in \Lambda$ such that $\lambda \geq \lambda_K$ implies $x_\lambda \in \tilde{X}_p \setminus K$.

We can now import the following results about the unified space.

Theorem C.3.6 ([Why66, §3]). *Let $p : X \rightarrow Y$ be a continuous map between locally compact Hausdorff spaces. Then:*

- (i) *the inclusion $X \hookrightarrow \tilde{X}_p$ is open, hence a homeomorphism onto its image;*
- (ii) *the inclusion $Y \hookrightarrow \tilde{X}_p$ is closed, hence a homeomorphism onto its image;*
- (iii) *\tilde{X}_p is locally compact and Hausdorff;*
- (iv) *$\tilde{p} : \tilde{X}_p \rightarrow Y$ is perfect;*
- (v) *\tilde{X}_p is second-countable if both X and Y are second-countable; and*
- (vi) *if X is compact then the topology on \tilde{X}_p is the disjoint union topology.*

Proof. All of these results, except for (v), are contained in Whyburn’s original papers [Why53; Why66]. For (v) choose a countable base \mathcal{B}_X for X and \mathcal{B}_Y for Y . Since X is σ -compact there is a sequence $\mathcal{K} := \{K_i\}_{i \in \mathbb{N}}$ of increasing compact subsets of X such that $X = \bigcup_{i \in \mathbb{N}} K_i$. Now for every compact $K \subseteq X$ there exists $i \in \mathbb{N}$ such that $K \subseteq K_i$. It follows that $\mathcal{B}_X \cup \{\tilde{p}^{-1}(V) \cap (\tilde{X}_p \setminus K) \mid V \in \mathcal{B}_Y, K \in \mathcal{K}\}$ is a countable base for \tilde{X}_p . \square

Example C.3.7. Let $X = \mathbb{R}^2$ and $Y = \mathbb{R}$ with the Euclidean topology. Let $p : X \rightarrow Y$ be given by $p(x, y) = x$. The unified space \tilde{X}_p is homeomorphic to the cylinder $\mathbb{R} \times S^1$ with $\tilde{p} : \mathbb{R} \times S^1 \rightarrow \mathbb{R}$ given by $\tilde{p}(x, y) = x$. Since \tilde{X}_p is not compact, the unified space is not a compactification. \triangle

In many of our examples $p : X \rightarrow Y$ is not surjective.

Example C.3.8. Let $Y = \mathbb{R}$ with the Euclidean topology, and $X = ((0, 1] \times [0, 1]) \cup ((1, 2] \times [0, 1])$ with the subspace topology inherited from \mathbb{R}^2 . Then X is locally compact and Hausdorff. Let $p : X \rightarrow Y$ denote the projection $p(x, y) = x$. The unified space can be described (up to homeomorphism) by Figure C.3. As in the previous example \tilde{X}_p is not compact. Visually, it is straightforward to verify that \tilde{p} is perfect. \triangle

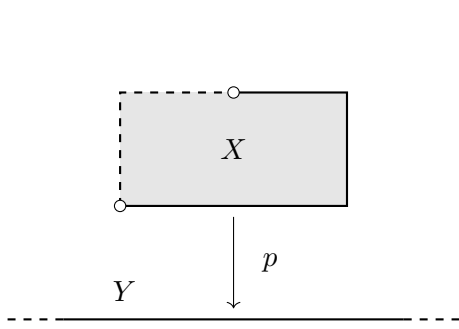


Figure C.2: The space X of Example C.3.8 with projection $p : X \rightarrow Y$.

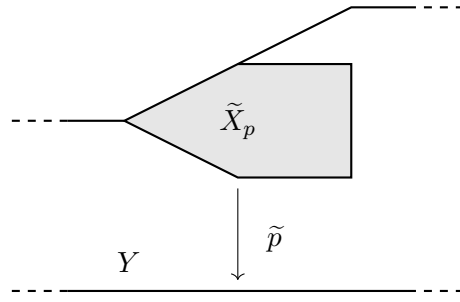


Figure C.3: An embedding of the unified space \tilde{X} of Example C.3.8 into \mathbb{R}^2 with the projection $\tilde{p} : \tilde{X}_p \rightarrow Y$ given mapping a point in \tilde{X}_p to the point directly below it in Y .

The fibres of \tilde{p} can be identified with the one-point compactifications of the fibres of p , where the one-point compactification of an already-compact space adds an isolated point. In particular, the one point compactification of the empty set is a singleton.

Lemma C.3.9. *For each $y \in Y$ the space $\tilde{p}^{-1}(y)$, equipped with the subspace topology from \tilde{X}_p , is homeomorphic to the one-point compactification of $p^{-1}(y)$, equipped with the subspace topology from X .*

Proof. As a set $\tilde{p}^{-1}(y) = p^{-1}(y) \cup \{y\}$. Since $p^{-1}(y)$ is closed in X , every compact subset of $p^{-1}(y)$ is compact in X . It follows—by restricting the basis \mathcal{B} to $\tilde{p}^{-1}(y)$ —that

$$\{U \mid U \text{ open in } p^{-1}(y)\} \cup \{\tilde{p}^{-1}(y) \setminus K \mid K \text{ compact in } X\}$$

is a basis for the subspace topology on $\tilde{p}^{-1}(y)$. This basis generates the one-point compactification topology on $\tilde{p}^{-1}(y)$ where y corresponds to the “point at infinity”. \square

We also record the following.

Lemma C.3.10. *A set $K \subseteq \widetilde{X}_p$ is compact if and only if K is closed and there is a compact subset K' of Y such that $K \subseteq \tilde{p}^{-1}(K')$.*

Proof. For the “only if” direction note that K is closed subset of $\tilde{p}^{-1}(\tilde{p}(K))$. For the “if” direction observe that K is a closed subset of the compact set $\tilde{p}^{-1}(K')$. \square

C.3.2 The minimal perfection

In general, \widetilde{X}_p is larger than it needs to be: if U is an open subset of $p(X)$ and the restriction of p to $p^{-1}(U)$ is already proper, then U could be excised from \widetilde{X}_p without preventing \tilde{p} from being proper. We now identify a closed subset of \widetilde{X}_p which gives rise to a minimal perfection of p . To this end define

$$X_p^+ := \overline{X} \cup \tilde{p}^{-1}\left(Y \setminus \text{int}(\overline{p(X)})\right) \subseteq \widetilde{X}_p. \quad (\text{C.2})$$

Equip X_p^+ with the subspace topology from \widetilde{X}_p . Then X_p^+ is closed in \widetilde{X}_p and hence locally compact. The projection map $p^+ : X_p^+ \rightarrow Y$ defined by the restriction of \tilde{p} to X_p^+ is again proper since X_p^+ is closed in \widetilde{X}_p . To see that p^+ is surjective first note that p^+ maps onto $Y \setminus \text{int}(\overline{p(X)})$. Since p^+ is closed, $p^+(\overline{X}) = \overline{p^+(X)}$. Then $\text{int}(\overline{p(X)}) \subseteq \overline{p(X)} = \overline{p^+(X)} = p^+(\overline{X})$. Consequently, p^+ is surjective.

Definition C.3.11. We call the pair (X_p^+, p^+) the *minimal perfection* of $p : X \rightarrow Y$.

Example C.3.12. Let X, Y and $p : X \rightarrow Y$ be the same as in Example C.3.8. Then $\overline{p(X)} = [0, 2]$, so that $Y \setminus \text{int}(\overline{p(X)}) = (-\infty, 0] \cup [2, \infty)$. The minimal perfection can be described by Figure C.4 up to homeomorphism. It is worth noting that if the point labelled x were omitted, then p^+ would no longer be proper. \triangle

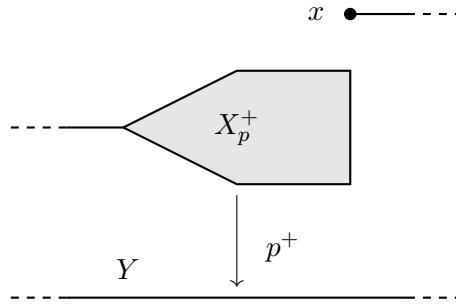


Figure C.4: An embedding of the minimal perfection X_p^+ of Example C.3.12 into \mathbb{R}^2 with the projection $p^+ : X_p^+ \rightarrow Y$ given mapping a point in X_p^+ to the point directly below it in Y .

In the case where p is surjective, the minimal perfection appeared in [Why53] and [AD14]; in the latter it was referred to as the Alexandroff fibrewise compactification of p . Indeed, if Y is a singleton, then (X_p^+, p^+) coincides with the Alexandroff (one-point) compactification of X . The following result justifies the term *minimal*.

Proposition C.3.13 (cf. [AD14, Proposition 1.10]). *Let $p : X \rightarrow Y$ be a continuous map between locally compact Hausdorff spaces. Then inside \widetilde{X}_p we have $\overline{X} \cap Y = Y \setminus U$*

where U is the largest open subset of Y such that the restriction of p to $p^{-1}(U)$ is proper. We can identify,

$$U = \{y \in Y \mid \text{there exists a precompact open neighbourhood } W \text{ of } y \\ \text{such that } p^{-1}(\overline{W}) \text{ is compact in } X\}.$$

Furthermore, X_p^+ is the smallest closed of \widetilde{X}_p containing X (in the inclusion ordering) such that the restriction of \tilde{p} to x_p^+ is perfect.

Proof. The existence of such an open set U follows from [AD14, Remark 1.7]. We first show that $X \sqcup (Y \setminus U)$ is closed in \widetilde{X}_p . Fix a compact set $K \subset Y \setminus (Y \setminus U) = U$. Then $p^{-1}(K)$ is compact since $p|_{p^{-1}(U)}$ is proper. It follows from Remark C.3.4 that $X \sqcup (Y \setminus U)$ is closed in \widetilde{X}_p , so $\overline{X} \subseteq X \sqcup (Y \setminus U)$. Consequently, $\overline{X} \cap Y \subseteq Y \setminus U$.

We now claim that $Y \setminus U \subseteq \overline{X}$. It suffices to show that $\widetilde{X}_p \setminus \overline{X} \subseteq U$. Since $\widetilde{X}_p \setminus \overline{X}$ is open in \widetilde{X}_p and contained in Y , the set $\widetilde{X}_p \setminus \overline{X}$ is open in Y . For any compact $K \subseteq \widetilde{X}_p \setminus \overline{X}$, the set $\tilde{p}^{-1}(K) \cap \overline{X} = p^{-1}(K) \cap X$ is compact in \widetilde{X}_p . In particular, $p^{-1}(K)$ is compact in X . Thus, $\widetilde{X}_p \setminus \overline{X} \subseteq U$ by maximality of U .

For the characterisation of U , suppose that $y \in U$. Since Y is locally compact and Hausdorff, we can find a precompact open neighbourhood W of y such that $\overline{W} \subseteq U$. Then $W \cap U$ is a precompact open neighbourhood of y with $\overline{W \cap U} \subseteq U$ such that $p^{-1}(\overline{W \cap U}) = (p|_{p^{-1}(U)})^{-1}(\overline{W \cap U})$ is compact.

On the other hand, suppose that $y \in Y$ and that W is a precompact open neighbourhood of y such that $p^{-1}(\overline{W})$ is compact. Fix a compact subset K of $p \circ p^{-1}(W) \subseteq W$. Then $(p|_{p^{-1}(W)})^{-1}(K)$ is a closed subset of $p^{-1}(\overline{W})$, and hence compact. It now follows from maximality of U that $W \subseteq U$, so that $y \in U$.

For the final statement, suppose that C is a closed subset of \widetilde{X}_p which contains X and that $\tilde{p}|_C$ is perfect. Then $\overline{X} \subseteq C$. On the other hand in order for $\tilde{p}|_C$ to be surjective, we require that $\tilde{p}^{-1}(Y \setminus \text{int}(\overline{p(X)})) \setminus \overline{X}$ is contained in C . Consequently, $X_p^+ = \bigcap \{C \text{ closed} \mid X \subseteq C \subseteq \widetilde{X}_p \text{ and } \tilde{p}|_C \text{ is perfect}\}$. \square

Remark C.3.14. Let U be as in Proposition C.3.13. We note that $\overline{p(X)}^c \subseteq U$ since $p^{-1}(Y \setminus \overline{p(X)}) = \emptyset$ is compact. In particular, $Y \setminus U \subseteq \overline{p(X)}$.

With U as in Proposition C.3.13, we write the minimal perfection of $p: X \rightarrow Y$ as

$$X_p^+ = X \sqcup \left(Y \setminus \left(U \cap \text{int} \left(\overline{p(X)} \right) \right) \right) \subseteq \widetilde{X}_p. \quad (\text{C.3})$$

Remark C.3.15. It is likely that the notion of perfection could be extended to a definition similar to [AD14, Definition 1.1] in the surjective case. This would allow for a notion of maximal/Stone-Ćech perfections. Similarly, there should be a characterisation in C^* -algebraic language as in [AD14, Proposition 1.2]. However, this is not required for our purposes.

To conclude this section, we have a result which helps us deal with the case when $p(X)$ is not closed.

Lemma C.3.16. *Suppose that $p: X \rightarrow Y$ is a continuous map between locally compact Hausdorff spaces. If $y \in \overline{p(X)} \setminus p(X)$ then for any precompact open neighbourhood W of y the set $p^{-1}(\overline{W})$ is not compact.*

Proof. Fix a precompact open neighbourhood W of $y \in \overline{p(X)} \setminus p(X)$, and suppose that $(y_\lambda)_{\lambda \in \Lambda}$ is a net in $p(X)$ such that $y_\lambda \rightarrow y$. Without loss of generality suppose that $y_\lambda \in W$ for all $\lambda \in \Lambda$. Choose $x_\lambda \in p^{-1}(y_\lambda) \subseteq p^{-1}(\overline{W})$. Suppose that (x_λ) admits a convergent subnet $(x_{\lambda'})_{\lambda' \in \Lambda'}$ for $\Lambda \subseteq \Lambda'$, which converges to some $x \in X$. Since $p(x_{\lambda'}) = y_{\lambda'} \rightarrow y$ it follows that $p(x) = y$ which is absurd since $p^{-1}(y) = \emptyset$. Thus, (x_λ) is a net in $p^{-1}(\overline{W})$ with no convergent subnet. \square

C.3.3 Perfections and G -equivariance

We now take a look at how the unified space and minimal perfection behave with respect to groupoid actions. Some basic notions from the theory of groupoids, groupoid actions, and the associated C^* -algebras can be found in Appendix B.

Let G be a locally compact Hausdorff groupoid and suppose that X and Y are locally compact Hausdorff G -spaces with moment maps m_X and m_Y , respectively. Suppose that $p: X \rightarrow Y$ is G -equivariant. We show that \widetilde{X}_p and X_p^+ are both G -spaces with G -action induced by the actions of G on X and Y . In the case where $p: X \rightarrow Y$ is surjective and $Y = G^{(0)}$, this problem was studied by Anantharaman-Delaroche [AD14] in the context of G -equivariant fibrewise compactifications.

Proposition C.3.17. *Let G be a locally compact Hausdorff groupoid and suppose that $p: X \rightarrow Y$ is a continuous G -equivariant map between locally compact Hausdorff G -spaces. Then the unified space \widetilde{X}_p is a G -space with moment map $\widetilde{m}: \widetilde{X} \rightarrow G^{(0)}$ given by*

$$\widetilde{m}(x) = \begin{cases} m_X(x) & \text{if } x \in X, \\ m_Y(x) & \text{if } x \in Y, \end{cases} \quad (\text{C.4})$$

and left G -action $\widetilde{\cdot}: G \times_{s, \widetilde{m}} \widetilde{X} \rightarrow \widetilde{X}$ given by,

$$\gamma \widetilde{\cdot} x = \begin{cases} \gamma \cdot x & \text{if } x \in X, \\ \gamma \cdot y & \text{if } x \in Y. \end{cases} \quad (\text{C.5})$$

Moreover, $\widetilde{p}: \widetilde{X}_p \rightarrow Y$ is G -equivariant, and X can be identified with an open G -invariant subset of \widetilde{X}_p .

Proof. We first claim that $\widetilde{m}: \widetilde{X} \rightarrow G^{(0)}$ is continuous. Recall the characterisation of convergence from Remark C.3.5. Fix a net $(x_\lambda)_{\lambda \in \Lambda}$ with $x_\lambda \rightarrow x$ in \widetilde{X} . If $x \in X$ then since X is open in \widetilde{X} there exists λ_0 such that $\lambda \geq \lambda_0$ that $x_\lambda \in X$. It now follows from continuity of m_X that $\lim_\lambda \widetilde{m}(x_\lambda) = \lim_{\lambda \geq \lambda_0} m_X(x_\lambda) = m_X(x) = \widetilde{m}(x)$.

If $x \in Y$ then $\widetilde{p}(x_\lambda) \rightarrow \widetilde{p}(x) = x$. If $x_\lambda \in X$, then $\widetilde{p}(x_\lambda) = p(x_\lambda)$. So G -invariance implies that $m_Y(p(x_\lambda)) = m_X(x_\lambda)$. If $x_\lambda \in Y$, then $\widetilde{p}(x_\lambda) = x_\lambda$. Since

$$\begin{aligned} \widetilde{m}(x_\lambda) &= \begin{cases} m_X(x_\lambda) & \text{if } x_\lambda \in X, \\ m_Y(x_\lambda) & \text{if } x_\lambda \in Y, \end{cases} \\ &= m_Y(\widetilde{p}(x_\lambda)), \end{aligned}$$

it follows from continuity of m_Y and \tilde{p} that $\lim_\lambda \tilde{m}(x_\lambda) = \lim_\lambda m_Y(\tilde{p}(x_\lambda)) = m_Y(x) = \tilde{m}(x)$. Thus, \tilde{m} is continuous. Since \tilde{m} is continuous and $G^{(0)}$ is Hausdorff we deduce that $G \times_{s, \tilde{m}} \tilde{X}$ is a closed subset of $G \times \tilde{X}$.

We claim that $\tilde{\cdot}: G \times_{s, \tilde{m}} \tilde{X} \rightarrow \tilde{X}$ is continuous. To this end fix a convergent net $(\gamma_\lambda, x_\lambda)_{\lambda \in \Lambda}$ in $G \times_{s, \tilde{m}} \tilde{X}$ with limit (γ, x) . We show that $\gamma_\lambda \tilde{\cdot} x_\lambda \rightarrow \gamma \tilde{\cdot} x$. First suppose that $x \in X$. Since X is open in \tilde{X} it follows that $G \times_{s, m_X} X$ is open in $G \times_{s, \tilde{m}} \tilde{X}$. In particular, since $x_\lambda \rightarrow x$ there exists $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $(\gamma_\lambda, x_\lambda) \in G \times_{s, m_X} X$. Consequently, the continuity of the action of G on X implies that $\lim_\lambda \gamma_\lambda \tilde{\cdot} x_\lambda = \lim_{\lambda \geq \lambda_0} \gamma_\lambda \cdot x_\lambda = \gamma \cdot x = \gamma \tilde{\cdot} x$.

Now suppose that $x \in Y$. Since $x_\lambda \rightarrow x$ we have $\tilde{p}(x_\lambda) \rightarrow \tilde{p}(x)$. If $x_\lambda \in X$ then G -invariance implies $p(\gamma_\lambda \cdot x_\lambda) = \gamma_\lambda \cdot p(x_\lambda)$. Hence,

$$\begin{aligned} \tilde{p}(\gamma_\lambda \tilde{\cdot} x_\lambda) &= \begin{cases} p(\gamma_\lambda \cdot x_\lambda) & \text{if } x_\lambda \in X, \\ \gamma_\lambda \cdot x_\lambda & \text{if } x_\lambda \in Y, \end{cases} \\ &= \gamma_\lambda \cdot \tilde{p}(x_\lambda). \end{aligned} \tag{C.6}$$

It now follows from continuity of \tilde{p} , and the action of G on Y , that $\lim_\lambda \tilde{p}(\gamma_\lambda \tilde{\cdot} x_\lambda) = \lim_\lambda \gamma_\lambda \cdot \tilde{p}(x_\lambda) = \gamma \cdot \tilde{p}(x) = \tilde{p}(\gamma \tilde{\cdot} x)$. Note that $\gamma \cdot \tilde{p}(x) = \tilde{p}(\gamma \tilde{\cdot} x)$ in way analogous to (C.6).

To see that $\gamma_\lambda \tilde{\cdot} x_\lambda \rightarrow \gamma \tilde{\cdot} x$ it remains to show that for any compact subset K of X there exists $\lambda_K \in \Lambda$ such that $\lambda \geq \lambda_K$ implies $\gamma_\lambda \tilde{\cdot} x_\lambda \in \tilde{X} \setminus K$. Suppose for contradiction that there exists a compact subset K of X such that for any $\lambda_0 \in \Lambda$ there exists $\lambda \geq \lambda_0$ with $\gamma_\lambda \tilde{\cdot} x_\lambda \in K$. By passing to a subnet we can assume that $\gamma_\lambda \tilde{\cdot} x_\lambda \in K$ for all $\lambda \in \Lambda$. Since $K \subset X$ we have $\gamma_\lambda \tilde{\cdot} x_\lambda = \gamma_\lambda \cdot x_\lambda$ for all $\lambda \in \Lambda$. Now fix a precompact open neighbourhood W of γ in G . Since $\gamma_\lambda \rightarrow \gamma$ there exists λ_W such that $\lambda \geq \lambda_W$ implies $\gamma_\lambda \in W$. Continuity of the action of G on X implies that $\overline{W^{-1} \cdot K} = \overline{W^{-1}} \cdot K$ is compact. For each $\lambda \geq \lambda_W$ we have $x_\lambda = \gamma_\lambda^{-1} \cdot (\gamma_\lambda \cdot x_\lambda)$ in $\overline{W^{-1}} \cdot K$. Consequently, $\overline{W^{-1}} \cdot K$ is a compact subset of X containing the net $(x_\lambda)_{\lambda \in \Lambda}$ never leaves. This contradicts $x_\lambda \rightarrow x$ in \tilde{X} . Thus, $\gamma_\lambda \tilde{\cdot} x_\lambda \rightarrow \gamma \tilde{\cdot} x$ in \tilde{X} , so $\tilde{\cdot}$ is continuous.

The G -invariance of X follows from the definition of the action, and G -equivariance of \tilde{p} follows from an argument similar to (C.6). \square

Corollary C.3.18. *Let G be a locally compact Hausdorff groupoid and suppose that $p: X \rightarrow Y$ is a continuous G -equivariant map between locally compact Hausdorff G -spaces. Then the minimal perfection X_p^+ is a G -space with moment map $m^+: X_p^+ \rightarrow G^{(0)}$ and left action given by the restriction of (C.4) and (C.5) to X_p^+ . Moreover, $p^+: X_p^+ \rightarrow Y$ is G -equivariant, and X is an open G -invariant subset of X_p^+ .*

Proof. Since X_p^+ is closed in \tilde{X}_p it follows m^+ is continuous. Moreover, $G \times_{s, m^+} X^+$ is a closed subset of $G \times_{s, \tilde{m}} \tilde{X}$, from which it follows that the action of G on X^+ is continuous. The remainder of the statement follows immediately. \square

We examine the special case where $Y = G^{(0)}$. This was considered in [AD14].

Corollary C.3.19. *Suppose that X is a G -space with moment map $m: X \rightarrow G^{(0)}$. Then*

$\widetilde{m}: \widetilde{X}_m \rightarrow G^{(0)}$ defines a moment map for a G -action $\widetilde{\cdot}: G \times_{s,\widetilde{m}} \widetilde{X}_m \rightarrow \widetilde{X}_m$ given by

$$\gamma \cdot x = \begin{cases} \gamma \cdot x & \text{if } x \in X, \\ r(\gamma) & \text{if } x \in G^{(0)}, \end{cases}$$

so that \widetilde{X}_m is a G -space with perfect moment map. The corresponding statement also holds for the minimal perfection.

Recall from Lemma B.4.7 that a G -equivariant map $p: X \rightarrow Y$ induces a homomorphism $p^\times: G \ltimes X \rightarrow G \ltimes Y$ by $p^\times(\gamma, x) = (\gamma, p(x))$. We can apply this to the maps \widetilde{p} and p^+ from Proposition C.3.17 and Corollary C.3.18, respectively.

Lemma C.3.20. *Let G be a second-countable locally compact Hausdorff groupoid and suppose that $p: X \rightarrow Y$ is a G -invariant map between second-countable locally compact Hausdorff G -spaces X and Y . If $p: X \rightarrow Y$ is perfect then so is $p^\times: G \ltimes X \rightarrow G \ltimes Y$. In particular, if $p: X \rightarrow Y$ is G -equivariant and continuous, then $\widetilde{p}^\times: G \ltimes \widetilde{X}_p \rightarrow G \ltimes Y$ and $p^{+\times}: G \ltimes X_p^+ \rightarrow G \ltimes Y$ are both perfect groupoid homomorphisms.*

Proof. Clearly p^\times is surjective if p is surjective. Now suppose that K is a compact subset of $G \ltimes Y$. Let $q_1: G \times Y \rightarrow G$ and $q_2: G \times Y \rightarrow Y$ denote the projections onto first and second factors of $G \times Y$, respectively. Since $G \times_{s,m} Y$ is closed in $G \times Y$, K is compact in $G \times Y$. Hence, $K_G := q_1(K)$ is compact in G and $K_Y := q_2(K)$ is compact in Y . Moreover, Lemma C.1.1 implies that $K_G \times_{s,m} K_Y$ is compact and contains K . Now $(p^\times)^{-1}(K_G \ltimes K_Y) = K_G \ltimes_{s,m} p^{-1}(K_Y)$ is a compact subset of $G \ltimes X$ which contains the closed set $(p^\times)^{-1}(K)$, so $(p^\times)^{-1}(K)$ is compact. Thus, p^\times is proper.

The final statement follows from Proposition C.3.17 and Corollary C.3.18. \square

APPENDIX D

Miscellanea

Proposition D.0.1. *Suppose that*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{q} & C & \longrightarrow & 0 \\ & & \downarrow \beta & & \downarrow \gamma & & \parallel & & \\ 0 & \longrightarrow & D & \xrightarrow{\delta} & X & \xrightarrow{p} & C & \longrightarrow & 0 \end{array}$$

is a commuting diagram of Abelian groups with exact rows. Then $X = \delta(D) + \gamma(B)$. If

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{q} & C & \longrightarrow & 0 \\ & & \downarrow \beta & & \downarrow \gamma' & & \parallel & & \\ 0 & \longrightarrow & D & \xrightarrow{\delta'} & X' & \xrightarrow{p'} & C & \longrightarrow & 0 \end{array}$$

is another commuting diagram with exact rows, then there is an isomorphism $\Upsilon : X \rightarrow X'$ satisfying $\Upsilon(\delta(d) - \gamma(b)) = \delta'(d) - \gamma'(b)$.

Proof. Fix $x \in X$, and choose $b \in B$ such that $q(b) = p(x)$. Then $p(x - \gamma(b)) = p(x) - q(b) = 0$. Hence, there exists $d \in D$ such that $x - \gamma(b) = \delta(d)$.

For the second statement, let $x = \delta(d) - \gamma(b) \in X$. We aim to define $\Upsilon(x) = \delta'(d) - \gamma'(b)$. To see that Υ is well-defined, suppose that $\delta(d) = \gamma(b)$. Then $0 = p(\gamma(b)) = q(b)$, so $b = \alpha(a)$ for some $a \in A$. It follows that $\delta(d) = \gamma(\alpha(a)) = \delta(\beta(a))$ so that $d = \beta(a)$. Now $\delta'(d) = \delta'(\beta(a)) = \gamma'(\alpha(a)) = \gamma'(b)$, so that Υ is well-defined.

To see that Υ is an isomorphism, consider the diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \xrightarrow{\delta} & X & \xrightarrow{p} & C & \longrightarrow & 0 \\ & & \parallel & & \downarrow \Upsilon & & \parallel & & \\ 0 & \longrightarrow & D & \xrightarrow{\delta'} & X' & \xrightarrow{p'} & C & \longrightarrow & 0 \end{array} \quad . \quad (D.1)$$

We have $\Upsilon(\delta(d)) = \delta'(d)$ for all $d \in D$ by construction of Υ , and for each $x = \delta(d) - \gamma(b)$ in X we have

$$p(x) = -p(\gamma(b)) = -q(b) = -p'(\gamma'(b)) = p'(\delta'(d) - \gamma'(b)) = p'(\Upsilon(x)).$$

Consequently, the diagram (D.1) commutes. It now follows from the Five Lemma that Υ is an isomorphism. \square

Remark D.0.2. If the Abelian groups and group homomorphisms of Proposition D.0.1 are replaced by C^* -algebras and $*$ -homomorphisms, then the map $\Upsilon: X \rightarrow X'$ is a $*$ -preserving vector space isomorphism. This follows from the definition of Υ and the $*$ -linearity of δ , γ , δ' , and γ' . However, Υ need not be a $*$ -homomorphism (see [Ped99, Example 5.4]). In certain cases—for example if $D = \overline{\beta(A)D}$ (see [Ped99, Theorem 2.4])— Υ is a $*$ -homomorphism, and hence an isomorphism of C^* -algebras.

The Splitting Lemma follows from Proposition D.0.1 by setting $B = C$, $A = 0$, $X' = C \oplus D$, and defining $\gamma' : C \rightarrow C \oplus D$ and $\delta' : D \rightarrow C \oplus D$ to be the canonical inclusions.

Corollary D.0.3 (The Splitting Lemma). *Suppose that*

$$0 \longrightarrow D \xrightarrow{\delta} X \xrightarrow{p} C \longrightarrow 0 \quad .$$

$\xleftarrow{\gamma}$

is a split exact sequence of Abelian groups. Then $X \cong D \oplus C$.

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